

Next we prove generalizations of Theorems 2.2 and 2.3 and Corollaries 2.2.1 and 2.3.1. These generalizations respectively are Theorems 2.4 and 2.5 and Corollaries 2.4.1 and 2.5.1. Theorem 2.4 states that

$$\text{Det}(\gamma * \Pi_1^0, \beta * \Sigma_1^0)^*$$

if $L(0^{\beta\#_{\gamma+1}^1})[\#_{\gamma}^1] \models$ “ $r^{\#_{\gamma}^1}$ exists for every real r ,” and Theorem 2.5 states that

$$\text{Det}(\gamma * \Pi_1^0, \beta * \Sigma_1^0)_+^*$$

follows from the existence of $0^{(\beta+1)\#_{\gamma+1}^1}$. The proof of Theorems 2.4 and 2.5 and Corollaries 2.4.1 and 2.5.1 is by induction on β . Theorems 2.2 and 2.3 and Corollaries 2.2.1 and 2.3.1 are these results for $\beta = 1$ and therefore constitute the Base Step for our inductive proof (with respect to β). We assume the following in the proofs of Theorems 2.4 and 2.5:

Induction Hypothesis. Assume $\beta\#_{\gamma+1}^1(0)$ exists. Then every $(\gamma * \Pi_1^0, (\beta - 1) * \Pi_1^0)_+^*$ game has a w.s. in $L(\beta\#_{\gamma+1}^1(0))$. Further suppose

$B_{\gamma}, B_{\gamma-1}, B_{\gamma-2}, \dots, B_1 \in \Pi_1^0, C_1, C_2, C_3, \dots, C_{\beta-1} \in \Sigma_1^0, \langle A_{\alpha} | \alpha < \omega^2 \rangle$, and $m \in \omega$

witness that A is $(\gamma * \Pi_1^0, (\beta - 1) * \Sigma_1^0)_+^*$. Let $s_{2\beta-1}^{\gamma}$ be a w.s. for the $G_{2\beta-1}^{\gamma}$

auxiliary game determined by

$$B_{\gamma}, B_{\gamma-1}, B_{\gamma-2}, \dots, B_1, C_1, C_2, C_3, \dots, C_{\beta-1}, \langle A_{\alpha} | \alpha < \omega^2 \rangle, \text{ and } m.$$

(See Theorem 2.5 for the definition of $G_{2\beta-1}^{\gamma}$.) Then A has a w.s. $s \in$

$L(\beta\#_{\gamma+1}^1(0))$ and s is a w.s. of the player for whom $s_{2\beta-1}^{\gamma}$ is a w.s.

Now we prove the most general results of this paper:

Theorem 2.4. If $L(0^{\beta\#_{\gamma+1}^1})[\#_{\gamma}^1] \models$ “ $r^{\#_{\gamma}^1}$ exists for every real r ,” then

$$\text{Det}(\gamma * \Pi_1^0, \beta * \Sigma_1^0)^*.$$

Proof: Assume $L(0^{\beta\#_{\gamma+1}^1})[\#_{\gamma}^1] \models$ “ $r^{\#_{\gamma}^1}$ exists for every real r .” Let

$$B_{\gamma}, B_{\gamma-1}, B_{\gamma-2}, \dots, B_1 \in \Pi_1^0, C_1, C_2, C_3, \dots, C_{\beta} \in \Sigma_1^0, \text{ and } \langle A_{\alpha} \mid \alpha < \omega^2 \rangle$$

strongly witness $A \in (\gamma * \Pi_1^0, \beta * \Sigma_1^0)^*$. Wlog $C_1 \subseteq C_2 \subseteq C_3 \subseteq \dots \subseteq C_{\beta}$. Then

for $1 \leq i \leq \beta$ and $1 \leq j \leq \gamma$, there exist R_{C_i} and R_{B_j} in Δ_1^0 such that

i.) $B_j(x, n) \leftrightarrow \forall k R_{B_j}(\bar{x}(k), n);$

ii.) $C_i(x, n) \leftrightarrow \exists k R_{C_i}(\bar{x}(k), n);$

iii.) if $\neg R_{B_j}(\bar{x}(k), n)$ and $\forall k' < k R_{B_j}(\bar{x}(k'), n)$, then k is odd; and

iv.) if $R_{C_i}(\bar{x}(k), n)$ and $\forall k' < k \neg R_{C_i}(\bar{x}(k'), n)$, then k is odd.

We show that G_A has a w.s. s . Conditions (iii) and (iv) help to simplify the proof.

We describe an open game $G_{2\beta}^{\gamma}$ which has a w.s. $s_{2\beta}^{\gamma} \in L(0^{\beta\#_{\gamma+1}^1})[\#_{\gamma}^1]$. We integrate $s_{2\beta}^{\gamma}$ to get the w.s. $s \in L(0^{\beta\#_{\gamma+1}^1})[\#_{\gamma}^1]$ for G_A . The game $G_{2\beta}^{\gamma}$ combines the Borel auxiliary moves Q_i and $\langle \hat{q}_i, q_i \rangle$ (for $1 \leq i \leq \beta$) of the game $G_{2\beta}^1$ (i.e. of the game $G^{2\beta}$ described in Section One) with the moves of G_1^{γ} .

The initial moves of $G_{2\beta}^{\gamma}$ are β Borel auxiliary moves Q_i and $\langle \hat{q}_i, q_i \rangle$:

$$G_{2\beta}^{\gamma} \quad \begin{array}{c} \text{I} \\ \text{II} \end{array} \quad \begin{array}{c} Q_{\beta} \\ \langle \hat{q}_{\beta}, q_{\beta} \rangle \end{array} \quad \begin{array}{c} Q_{\beta-1} \\ \langle \hat{q}_{\beta-1}, q_{\beta-1} \rangle \end{array} \quad \begin{array}{c} Q_{\beta-2} \\ \langle \hat{q}_{\beta-2}, q_{\beta-2} \rangle \end{array} \quad \cdots \quad \begin{array}{c} Q_1 \\ \langle \hat{q}_1, q_1 \rangle \end{array} \quad \cdots$$

Each pair Q_i and $\langle \hat{q}_i, q_i \rangle$ is determined by the Σ_1^0 set $\{m \in \omega \mid \exists n R_{C_i}(m, n)\}$ and the game G_A . Furthermore, player I may only play $Q_i \in L(0^{\beta\#_{\gamma+1}^1})[\#_{\gamma}^1]$ (and in certain cases, we have stricter requirements for which Q_i may be played).

Let $\vec{Q} = \langle Q_i | \hat{q}_i = 1 \rangle$ and $\vec{q} = \langle q_i | \hat{q}_i = 0 \rangle$. Whenever

$$(Q_\beta; \langle \hat{q}_\beta, q_\beta \rangle; Q_{\beta-1}; \langle \hat{q}_{\beta-1}, q_{\beta-1} \rangle; Q_{\beta-2}; \langle \hat{q}_{\beta-2}, q_{\beta-2} \rangle; \dots; Q_i; \langle \hat{q}_i, q_i \rangle)$$

is a legal position in $G_{2\beta}^\gamma$, II must play so that

$$\text{v.)} \quad 1 \leq i < j \leq \beta \text{ and } \hat{q}_j = 1 \Rightarrow \hat{q}_i = 1.$$

We do not allow II to play $\hat{q}_i = 0$ if $i < j$ and $\hat{q}_j = 1$ since $C_i \subseteq C_j$. The moves of $G_{2\beta}^\gamma$ remaining after II plays $\langle \hat{q}_1, q_1 \rangle$ are the moves of

$$G_1^\gamma(\vec{Q}; \vec{q}).$$

Therefore, $G_{2\beta}^\gamma$ contains a sequence

$$\langle (T_k^1; \langle \hat{t}_k^1, t_k^1 \rangle) | k \in \omega \ \& \ \forall j < k \ \hat{t}_j^1 = 0 \rangle$$

of Borel auxiliary moves, which is related to the Π_1^0 set B_1 via R_{B_1} . In general,

if $1 \leq \delta \leq \gamma$ and $G_{2\beta}^\gamma$ contains a sequence

$$\langle (T_k^i; \langle \hat{t}_k^i, t_k^i \rangle) | k \in \omega \ \& \ \forall j < k \ \hat{t}_j^i = 0 \rangle$$

such that some $\hat{t}_{k(i)}^i = 1$ whenever $1 \leq i < \delta$, then $G_{2\beta}^\gamma$ contains a sequence

$$\langle (T_k^\delta; \langle \hat{t}_k^\delta, t_k^\delta \rangle) | k \in \omega \ \& \ \forall j < k \ \hat{t}_j^\delta = 0 \rangle$$

of Borel auxiliary moves, which is related to the Π_1^0 set B_δ via R_{B_δ} . Let

$$\vec{T}_\delta = \langle T_i^j | 1 \leq j \leq \delta \ \& \ \hat{t}_i^j = 1 \rangle, \vec{t}_\delta = \langle t_i^j | 1 \leq j \leq \delta \ \& \ \hat{t}_i^j = 0 \rangle,$$

$$\vec{T} = \vec{T}_\gamma, \text{ and } \vec{t} = \vec{t}_\gamma.$$

Therefore, whenever $\langle \hat{t}_{k(j)}^j, t_{k(j)}^j \rangle = \langle 1, - \rangle$ for $1 \leq j \leq \delta$,

$$\vec{T}_\delta = \langle T_{k(j)}^j | 1 \leq j \leq \delta \rangle \text{ and } \vec{t}_\delta = \langle t_i^j | 1 \leq j \leq \delta \ \& \ 0 \leq i < k(j) \rangle.$$

Now we describe the game $G_{2\beta}^\gamma$ for the two cases $\hat{q}_\beta = 1$ and $\hat{q}_\beta = 0$ separately:

Case 1. II plays $\hat{q}_\beta = 0$ in $G_{2\beta}^\gamma$: $\begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} Q_\beta \\ \langle 0, q_\beta \rangle \end{array} \cdot \cdot \cdot$

Player I must play $Q_{\beta-1} \in L(0^{(\beta-1)\#_{\gamma+1}^1})[\#_\gamma^1]$. In general, if player II plays $\hat{q}_j = 0$ for each $j > i$, then player I must play $Q_i \in L(0^{i\#_{\gamma+1}^1})[\#_\gamma^1]$. Let μ be least such that $\hat{q}_\mu = 0$. Then by (v), II must play so that $\hat{q}_j = 1$ iff $j < \mu$. For $j < \mu$, I must play $Q_j \in L(0^{(\mu-1)\#_{\gamma+1}^1})[\#_\gamma^1]$. Furthermore,

$$\vec{Q} = \langle Q_j | 1 \leq j < \mu \rangle \text{ and } \vec{q} = \langle q_j | \mu \leq j \leq \beta \rangle.$$

After II plays $\langle \hat{q}_1, q_1 \rangle$, the remaining moves of $G_{2\beta}^\gamma$ are the moves of $G_1^\gamma(\vec{Q}; \vec{q})$.

Whenever II plays $\hat{t}_{k(j)}^j = 1$ for $j \leq \delta$ and $\hat{t}_j^{\delta+1} = 0$ for all $j \in \omega$, the play of $G_{2\beta}^\gamma$ is

$$\begin{array}{c} G_{2\beta}^\gamma \quad \begin{array}{c} \text{I} \\ \text{II} \end{array} \quad \begin{array}{c} Q_\beta \\ \langle 0, q_\beta \rangle \end{array} \quad \begin{array}{c} Q_{\beta-1} \\ \langle \hat{q}_{\beta-1}, q_{\beta-1} \rangle \end{array} \quad \begin{array}{c} Q_{\beta-2} \\ \langle \hat{q}_{\beta-2}, q_{\beta-2} \rangle \end{array} \quad \dots \\ \dots \quad \begin{array}{c} Q_1 \\ \langle \hat{q}_1, q_1 \rangle \end{array} \quad \begin{array}{c} T_0^1 \\ \langle 0, t_0^1 \rangle \end{array} \quad \begin{array}{c} x(0), \lambda_0^0 \\ x(1), \lambda_1^0 \end{array} \quad \begin{array}{c} T_1^1 \\ \langle 0, t_1^1 \rangle \end{array} \quad \begin{array}{c} x(2), \lambda_2^0 \\ x(3), \lambda_3^0 \end{array} \quad \dots \\ \dots \quad \begin{array}{c} T_{k(1)-1}^1 \\ \langle 0, t_{k(1)-1}^1 \rangle \end{array} \quad \begin{array}{c} x(2k(1)-2), \lambda_{2k(1)-2}^0 \\ x(2k(1)-1), \lambda_{2k(1)-1}^0 \end{array} \quad \begin{array}{c} T_{k(1)}^1 \\ \langle 1, - \rangle \end{array} \\ T_0^2 \quad \begin{array}{c} x(2k(1)), \lambda_0^1 \\ \langle 0, t_0^2 \rangle \end{array} \quad \begin{array}{c} T_1^2 \\ x(2k(1)+1), \lambda_1^1 \end{array} \quad \begin{array}{c} x(2k(1)+2), \lambda_2^1 \\ \langle 0, t_1^2 \rangle \end{array} \quad \begin{array}{c} x(2k(1)+3), \lambda_3^1 \\ \dots \end{array} \\ \dots \\ \dots \quad \begin{array}{c} T_{k(\delta)-1}^\delta \\ \langle 0, t_{k(\delta)-1}^\delta \rangle \end{array} \quad \begin{array}{c} x(2k(1)\dots+2k(\delta)-2), \lambda_{2k(\delta)-2}^{\delta-1} \\ x(2k(1)\dots+2k(\delta)-1), \lambda_{2k(\delta)-1}^{\delta-1} \end{array} \quad \begin{array}{c} T_{k(\delta)}^\delta \\ \langle 1, - \rangle \end{array} \\ T_0^{\delta+1} \quad \begin{array}{c} x(2k(1)\dots+2k(\delta)), \lambda_0^\delta \\ \langle 0, t_0^{\delta+1} \rangle \end{array} \quad \begin{array}{c} T_1^{\delta+1} \\ x(2k(1)\dots+2k(\delta)+1), \lambda_1^\delta \end{array} \quad \begin{array}{c} x(2k(1)\dots+2k(\delta)+2), \lambda_2^\delta \\ \langle 0, t_1^{\delta+1} \rangle \end{array} \quad \begin{array}{c} x(2k(1)\dots+2k(\delta)+3), \lambda_3^\delta \\ \dots \end{array} \\ \dots \quad \begin{array}{c} T_i^{\delta+1} \\ \langle 0, t_i^{\delta+1} \rangle \end{array} \quad \begin{array}{c} x(2k(1)\dots+2k(\delta)+2i), \lambda_{2i}^\delta \\ x(2k(1)\dots+2k(\delta)+2i+1), \lambda_{2i+1}^\delta \end{array} \quad \dots \end{array}$$

Whenever $\hat{t}_{k(\delta)}^\delta = 1$ for $1 \leq \delta \leq \gamma$, the play of $G_{2\beta}^\gamma$ is

$$\begin{array}{c} G_{2\beta}^\gamma \quad \begin{array}{c} \text{I} \\ \text{II} \end{array} \quad \begin{array}{c} Q_\beta \\ \langle 0, q_\beta \rangle \end{array} \quad \begin{array}{c} Q_{\beta-1} \\ \langle \hat{q}_{\beta-1}, q_{\beta-1} \rangle \end{array} \quad \begin{array}{c} Q_{\beta-2} \\ \langle \hat{q}_{\beta-2}, q_{\beta-2} \rangle \end{array} \quad \dots \\ \dots \quad \begin{array}{c} Q_1 \\ \langle \hat{q}_1, q_1 \rangle \end{array} \quad \begin{array}{c} T_0^1 \\ \langle 0, t_0^1 \rangle \end{array} \quad \begin{array}{c} x(0), \lambda_0^0 \\ x(1), \lambda_1^0 \end{array} \quad \begin{array}{c} T_1^1 \\ \langle 0, t_1^1 \rangle \end{array} \quad \begin{array}{c} x(2), \lambda_2^0 \\ x(3), \lambda_3^0 \end{array} \quad \dots \\ \dots \quad \begin{array}{c} T_{k(1)-1}^1 \\ \langle 0, t_{k(1)-1}^1 \rangle \end{array} \quad \begin{array}{c} x(2k(1)-2), \lambda_{2k(1)-2}^0 \\ x(2k(1)-1), \lambda_{2k(1)-1}^0 \end{array} \quad \begin{array}{c} T_{k(1)}^1 \\ \langle 1, - \rangle \end{array} \end{array}$$

$$\begin{array}{ccccccc}
T_0^2 & x(2k(1)), \lambda_0^1 & T_1^2 & x(2k(1)+2), \lambda_2^1 & \dots & & \\
\langle 0, t_0^2 \rangle & x(2k(1)+1), \lambda_1^1 & \langle 0, t_1^2 \rangle & x(2k(1)+3), \lambda_3^1 & & & \\
\vdots & & & & & & \\
& & \dots & T_{k(\delta)-1}^\delta & x(2k(1)\cdots+2k(\delta)-2), \lambda_{2k(\delta)-2}^{\delta-1} & T_{k(\delta)}^\delta & \\
& & & \langle 0, t_{k(\delta)-1}^\delta \rangle & x(2k(1)+\cdots+2k(\delta)-1), \lambda_{2k(\delta)-1}^{\delta-1} & \langle 1, - \rangle & \\
T_0^{\delta+1} & x(2k(1)+\cdots+2k(\delta)), \lambda_0^\delta & T_1^{\delta+1} & x(2k(1)+\cdots+2k(\delta)+2), \lambda_2^\delta & \dots & & \\
\langle 0, t_0^{\delta+1} \rangle & x(2k(1)+\cdots+2k(\delta)+1), \lambda_1^\delta & \langle 0, t_1^{\delta+1} \rangle & x(2k(1)+\cdots+2k(\delta)+3), \lambda_3^\delta & & & \\
\vdots & & & & & & \\
& & \dots & T_{k(\gamma)-1}^\gamma & x(2k(1)+\cdots+2k(\gamma)-2), \lambda_{2k(\gamma)-2}^{\gamma-1} & T_{k(\gamma)}^\gamma & \\
& & & \langle 0, t_{k(\gamma)-1}^\gamma \rangle & x(2k(1)+\cdots+2k(\gamma)-1), \lambda_{2k(\gamma)-1}^{\gamma-1} & \langle 1, - \rangle & \\
x(2k(1)+\cdots+2k(\gamma)), \xi_0 & x(2k(1)+\cdots+2k(\gamma)+2), \xi_2 & \dots & & & & \\
x(2k(1)+\cdots+2k(\gamma)+1), \xi_1 & x(2k(1)+\cdots+2k(\gamma)+3), \xi_3 & & & & &
\end{array}$$

Player I may only play $T_i^1 \in L(\vec{Q})[\#_\gamma^1]$. If $\hat{t}_{k(j)}^j = 1$ for $j < \delta$, then $\vec{T}_j = \langle T_{k(i)}^i | i \leq j \rangle$ for $j < \delta$ and player I may only play $T_i^\delta \in L(\vec{Q}, \vec{T}_{\delta-1})[\#_{\gamma+1-\delta}^1]$.

Let $k(0)$ be least such that

$$R_{C_\mu}(q_\mu, k(0)) \text{ and } \forall j < k(0) \neg R_{C_\mu}(q_\mu, j).$$

Any λ_i^0 's played must be properly ordered with respect to

$$\langle A_\alpha | \alpha < \omega \cdot (k(0) + 1) \rangle \text{ using } \langle \omega_{i+1}^{L(\#_{\gamma+1}^1(\vec{Q}))} | i \leq k(0) \rangle.$$

(Notice $L(\#_{\gamma+1}^1(\vec{Q}))$ exists for this case since $\vec{Q} \in L(0^{(\beta-1)\#_{\gamma+1}^1})[\#_\gamma^1]$.) In

general, for $\delta \geq 0$, any λ_i^δ 's played are properly ordered with respect to

$$\langle A_\alpha | \alpha < \omega \cdot (k(\delta) + 1) \rangle \text{ using } \langle \omega_{i+1}^{L(\#_{\gamma-\delta+1}^1(\vec{Q}, T_{k(1)}^1, T_{k(2)}^2, \dots, T_{k(\delta)}^\delta))} | i \leq k(\delta) \rangle.$$

Any ξ_i 's played must be properly ordered with respect to

$$\langle A_\alpha | \alpha < \omega \cdot (k(\gamma) + 1) \rangle \text{ using } \langle \omega_{i+1}^{L(\#_1^1(\vec{Q}, T_{k(1)}^1, T_{k(2)}^2, \dots, T_{k(\gamma)}^\gamma))} | i \leq k(\gamma) \rangle.$$

Case 2. II plays $\hat{q}_\beta = 1$ in $G_{2\beta}^\gamma$: $\begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} Q_\beta \\ \langle 1, - \rangle \end{array} \dots$

By (v), II may only play $\hat{q}_i = 1$ and I may only $Q_i \in L(0^{\beta\#_{\gamma+1}^1})[\#_\gamma]$:

$$G_{2\beta}^\gamma \begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} Q_\beta \\ \langle 1, - \rangle \end{array} \begin{array}{c} Q_{\beta-1} \\ \langle 1, - \rangle \end{array} \begin{array}{c} Q_{\beta-2} \\ \langle 1, - \rangle \end{array} \dots \begin{array}{c} Q_1 \\ \langle 1, - \rangle \end{array} \dots$$

as in Case 1 (except that $\vec{Q} = \langle Q_i | 1 \leq i \leq \beta \rangle$ here): If $\hat{t}_{k(j)}^j = 1$ for all $j < \delta$, then $\vec{T}_j = \langle T_{k(i)}^i | 1 \leq i \leq j \rangle$ for all $j < \delta$, (for $\delta \geq 1$) player I may only play $T_i^\delta \in L(\vec{Q}, \vec{T}_{\delta-1})[\#_{\gamma+1-\delta}^1]$, and (for $\delta \geq 2$) any $\lambda_i^{\delta-1}$ played must be properly ordered with respect to

$$\langle A_\alpha | \alpha < \omega \cdot (k(\delta - 1) + 1) \rangle \text{ using } \langle \omega_{i+1}^{L(\#_{\gamma+2-\delta}^1(\vec{Q}, \vec{T}_{\delta-1}))} | i \leq k(\delta - 1) \rangle.$$

Any ξ_i 's played must be properly ordered with respect to

$$\langle A_\alpha | \alpha < \omega \cdot (k(\gamma) + 1) \rangle \text{ using } \langle \omega_{i+1}^{L(\#_1^1(\vec{Q}, \vec{T}_\gamma))} | i \leq k(\gamma) \rangle.$$

Player I wins $G_{2\beta}^\gamma$ iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move. $G_{2\beta}^\gamma$ is an open game and therefore we define, for each ordinal α , P_α as the set of positions with ordinal α and let $P = \bigcup_{\alpha \in ON} P_\alpha$. If p is a legal position in $G_{2\beta}^\gamma$, let ℓ_p denote the set of legal positions in $G_{2\beta}^\gamma$ consistent with p . The set ℓ of legal positions for $G_{2\beta}^\gamma$ is in $L(0^{\beta\#_{\gamma+1}^1})[\#_\gamma^1]$.

Using $\langle P_\alpha | \alpha \in ON \rangle$ and Lemma 0.14, define a wellordering \prec of ℓ and the canonical w.s. $s_{2\beta}^\gamma$ for $G_{2\beta}^\gamma$ so that Lemma 2.4.1 (below) and the following hold: $s_{2\beta}^\gamma$ is in $L(0^{\beta\#_{\gamma+1}^1})[\#_\gamma^1]$, and if p is a legal position in $G_{2\beta}^\gamma$, then $s_{2\beta}^\gamma|_{\ell_p}$ is a w.s. for $(G_{2\beta}^\gamma)_p$ and is definable in any inner model of ZF in which $\prec|_{\ell_p}$ is definable. Analogous to Lemmas 1.6.1 and 2.2.1, $s_{2\beta}^\gamma$ has the following properties:

Lemma 2.4.1. Let p be a legal position in $G_{2\beta}^\gamma$. Let $\#_0^1(U) = U$ for any U .

Then $s_{2\beta}^\gamma|_{\ell_p}$ is a w.s. for $(G_{2\beta}^\gamma)_p$ and has the following properties:

vi.) If $1 \leq \delta \leq \beta$, position p includes the moves $\langle \hat{q}_i, q_i \rangle = \langle 0, q_i \rangle$ for $\delta \leq i \leq \beta$, and μ is least such that $R_{C_\delta}(q_\delta, \mu)$, then $s_{2\beta}^\gamma | \ell_p$ is definable in $L((\delta - 1)\#_{\gamma+1}^1(0))[\#_\gamma^1]$ from $\langle \omega_{i+1}^{L(\delta\#_{\gamma+1}^1(0))} | i \leq \mu \rangle$. In particular, if p includes the move $\langle 0, q_\beta \rangle$ and n is least such that $R_{C_\beta}(q_\beta, n)$, then $s_{2\beta}^\gamma | \ell_p$ is definable in $L((\beta - 1)\#_{\gamma+1}^1(0))[\#_\gamma^1]$ from $\langle \omega_{i+1}^{L(\beta\#_{\gamma+1}^1(0))} | i \leq n \rangle$.

vii.) If p includes $\langle \hat{q}_i, q_i \rangle = \langle 1, - \rangle$ for $1 \leq i \leq \beta$, then $s_{2\beta}^\gamma | \ell_p$ is definable in $L(\vec{Q})[\#_\gamma^1]$.

viii.) If $\delta \geq 1$ and p includes the moves $\langle \hat{t}_{k(j)}^j, t_{k(j)}^j \rangle = \langle 1, - \rangle$ for $1 \leq j \leq \delta$, then $s_{2\beta}^\gamma | \ell_p$ is definable in $L(\vec{Q}, \vec{T}_\delta)[\#_{\gamma-\delta}^1]$ from $\langle \omega_{i+1}^{L(\#_{\gamma+1-\delta}^1(\vec{Q}, \vec{T}_\delta))} | i \leq k(\delta) \rangle$. In particular, if p includes the moves $\langle \hat{t}_{k(\gamma)}^\gamma, t_{k(\gamma)}^\gamma \rangle = \langle 1, - \rangle$, then $s_{2\beta}^\gamma | \ell_p$ is definable in $L(\vec{Q}, \vec{T})$ from $\langle \omega_{i+1}^{L(\#_1^1(\vec{Q}, \vec{T}))} | i \leq k(\gamma) \rangle$.

By Lemma 2.4.1 and Corollaries 2.2.1 and 2.3.1, we have

Lemma 2.4.2. Let $q = (Q_\beta; \langle \hat{q}_\beta, q_\beta \rangle; Q_{\beta-1}; \langle \hat{q}_{\beta-1}, q_{\beta-1} \rangle; \dots; Q_1; \langle \hat{q}_1, q_1 \rangle)$

be a legal position of $G_{2\beta}^\gamma$. Let $1 \leq \delta \leq \gamma$ and let p_δ be a legal position of $G_{2\beta}^\gamma$ which extends q and whose last move is $\langle \hat{t}_{k(\delta)}^\delta, t_{k(\delta)}^\delta \rangle = \langle 1, - \rangle$. If

$$A' = A(\vec{Q}, \vec{T}_\delta; \vec{q}, \vec{t}_\delta, \bar{x}(2k_1 + \dots + 2k_\delta)),$$

then the following hold:

ix.) If $\hat{q}_\beta = 1$, then $\vec{Q} = \langle Q_i | 1 \leq i \leq \beta \rangle$ and the w.s. $s_{2\beta}^\gamma | \ell_q$ can be integrated so as to obtain a w.s.

$$s_q \in L(\vec{Q})[\#_\gamma^1] \text{ for } A(\vec{Q}).$$

If $\hat{q}_\beta = 0$, then the w.s. $s_{2\beta}^\gamma | \ell_q$ can be integrated so as to obtain a w.s.

$$s_q \in L(\#_{\gamma+1}^1(\vec{Q})) \text{ for } A(\vec{Q}; \vec{q}).$$

In either case, s_q is a w.s. of the player for whom $s_{2\beta}^\gamma$ is a w.s.

x.) The w.s. $s_{2\beta}^\gamma | \ell_{p_\delta}$ can be integrated so as to obtain a w.s. $s' \in L(\#_{\gamma-\delta+1}^1(\vec{Q}, \vec{T}_\delta))$ for A' and s' is a w.s. of the player for whom $s_{2\beta}^\gamma$ is a w.s.

xi.) Suppose $s_{2\beta}^\gamma$ is a w.s. for I. If p is a position consistent with s_q and the moves in p of player II are consistent with \vec{q} , then $p \in Q_i$ for i such that $\hat{q}_i = 1$. Therefore, if x is a play consistent with s_q , then $x \in A(\vec{q})$. If p is a position consistent with s' and the moves in p of player II are consistent with \vec{q} , \vec{t}_δ , and $\bar{x}(2k_1 + 2k_2 + 2k_3 + \cdots + 2k_\delta)$, then

$$p \in Q_i \text{ for } i \text{ such that } \hat{q}_i = 1$$

$$\text{and } p \in T_{k(j)}^j \text{ for } 1 \leq j \leq \delta.$$

If x is a play consistent with s' , then $x \in A(\vec{q}, \vec{t}_\delta, \bar{x}(2k_1 + \cdots + 2k_\delta))$.

xii.) Let $p = (x(0); x(1); x(2); \dots; x(i))$ be a position consistent with \vec{Q} . If p is consistent with s_q and if the moves in p of the player for whom s_q is not a w.s. are consistent with \vec{q} , then p is consistent with \vec{q} . If p is consistent with s' , \vec{Q} , and \vec{T}_δ , and if the moves in p of the player for whom s' is not a w.s. are consistent with \vec{q} , \vec{t}_δ , and $\bar{x}(2k_1 + 2k_2 + \cdots + 2k_\delta)$, then p is consistent with \vec{q} , \vec{t}_δ , and $\bar{x}(2k_1 + 2k_2 + \cdots + 2k_\delta)$.

By the Induction Hypothesis, we have the following:

Lemma 2.4.3. If $q = (Q_\beta, \langle 0, q_\beta \rangle)$ is a legal position of $G_{2\beta}^\gamma$, then $A(q_\beta)$ has

a w.s. $s' \in L(\beta \#_{\gamma+1}^1(0))$ such that:

(1) s' is a w.s. of the player for whom $s_{2\beta}^\gamma$ is a w.s.

(2) If p is a position of the game G_A such that p is consistent with s' and the moves in p of the player for whom s' is not a w.s. are consistent with q_β , then p is consistent with q_β .

Claim I: Player I has a w.s. for G_A if he has one for $G_{2\beta}^\gamma$.

Let's first consider the case in which $\langle \rangle \in P$. Then $s_{2\beta}^\gamma \in L(0^{\beta \#_{\gamma+1}^1})[\#_\gamma^1]$ is a w.s. for I in $G_{2\beta}^\gamma$. We use $s_{2\beta}^\gamma$ to define a w.s. s for I in G_A . For $1 \leq i \leq \beta$, let

$$Q_i = s_{2\beta}^\gamma(Q_\beta; \langle 1, - \rangle; Q_{\beta-1}; \langle 1, - \rangle; \dots; Q_{i+1}; \langle 1, - \rangle), \quad \langle \hat{q}_i, q_i \rangle = \langle 1, - \rangle,$$

$$\text{and } p_0 = (Q_\beta; \langle 1, - \rangle; Q_{\beta-1}; \langle 1, - \rangle; \dots; Q_1; \langle 1, - \rangle).$$

Then $\vec{Q} = \langle Q_i | 1 \leq i \leq \beta \rangle$. By Lemma 2.4.1(vii), $s_{2\beta}^\gamma|_{\ell_{p_0}}$ is a w.s. for $(G_{2\beta}^\gamma)_{p_0}$ and is definable in $L(\vec{Q})[\#_\gamma^1]$. By Lemma 2.4.2(ix), obtain a w.s. $s_0 \in L(\vec{Q})[\#_\gamma^1]$ for $A(\vec{Q})$ by integrating the w.s. $s_{2\beta}^\gamma|_{\ell_{p_0}}$ for $A(\vec{Q})$, and let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); \dots; x(i-1))$ such that $\forall i' \leq i \forall n \neg R_{C_j}(\bar{x}(i'), n)$ for $1 \leq j \leq \beta$.

Suppose we reach a position such that $\exists i \exists j \exists n R_{C_j}(\bar{x}(i), n)$. Since $C_j \subseteq C_\beta$, there is a least i such that $\exists n R_{C_\beta}(\bar{x}(i), n)$. By (iv), i is odd. Let $Q_\beta = s(\cdot)$ (as above), $\langle \hat{q}_\beta, q_\beta \rangle = \langle 0, \bar{x}(i) \rangle$, and $q = (Q_\beta; \langle \hat{q}_\beta, q_\beta \rangle)$. By Lemma 2.4.3, player I has a w.s. $s_1 \in L(\beta \#_{\gamma+1}^1(0))$ for $A(q_\beta)$. Let $s(p) = s_1(p)$ for any position p which extends $(x(0); x(1); x(2); \dots; x(i-1))$.

Claim: The strategy s of player I is a w.s. in G_A .

Suppose x is a play of G_A consistent with s . By the definition of s , there exists a w.s. $s_0 \in L(\vec{Q})[\#_\gamma^1]$ for $A(\vec{Q})$ and $x(2i) = s_0(x(1); x(3); \dots; x(2i-1))$ whenever $\forall i' \leq 2i \forall n \neg R_{C_j}(\bar{x}(i'), n)$ for $1 \leq j \leq \beta$. If $\forall i \forall n \neg R_{C_j}(\bar{x}(i), n)$ holds (for $1 \leq j \leq \beta$), then $x \in A(\vec{Q})$ so that $x \in A$ by Lemma 2.4.2(xi).

Otherwise, $\exists n R_{C_\beta}(\bar{x}(i), n)$ for some least i . By the definition of s , there exists a w.s. $s_1 \in L(\beta \#_{\gamma+1}^1(0))$ for $A(\bar{x}(i))$ such that

$$x(2k) = s_1(x(1); x(3); \dots; x(2k-1)) \text{ whenever } 2k > i.$$

Therefore, $x \in A(\bar{x}(i))$ so that $x \in A$. Thus, x is a win for I.

Claim II: Player II has a w.s. for G_A if he has one for $G_{2\beta}^\gamma$.

Now let's consider the case $\langle \rangle \notin P$. We integrate II's w.s. $s_{2\beta}^\gamma$ for $G_{2\beta}^\gamma$ to get the w.s. $s \in L(0^{\beta \#_{\gamma+1}^1})[\#_\gamma^1]$ for II in G_A . Let

$$Q_\beta = \{\text{positions } q \text{ in } G_A \mid \forall Q' \in L(\beta \#_{\gamma+1}^1(0))[\#_\gamma^1] \langle 0, q \rangle \neq s_{2\beta}^\gamma(Q')\}$$

and $Q_i = Q_\beta$ for $1 \leq i < \beta$. Clearly $\langle 1, - \rangle = s_{2\beta}^\gamma(Q_\beta)$. Furthermore, for $1 \leq i \leq \beta$,

$$Q_i \in L(0^{\beta \#_{\gamma+1}^1})[\#_\gamma^1], \text{ and by (v), } \langle 1, - \rangle = s_{2\beta}^\gamma(Q_\beta; Q_{\beta-1}; Q_{\beta-2}; \dots; Q_i).$$

Let $p_0 = (Q_\beta; \langle 1, - \rangle; Q_{\beta-1}; \langle 1, - \rangle; Q_{\beta-2}; \langle 1, - \rangle; \dots; Q_1; \langle 1, - \rangle)$. By Lemma 2.4.2(ix), integrate $s_{2\beta}^\gamma|_{\ell_{p_0}}$ so as to obtain a w.s. $s_0 \in L(\vec{Q})[\#_\gamma^1]$ for the game $A(\vec{Q})$. Let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); x(2); \dots; x(i-1))$ such that $\forall i' \leq i \bar{x}(i') \in Q_\beta = \bigcap_{1 \leq j \leq \beta} Q_j$.

Suppose we reach a position $\bar{x}(i)$ (of least length) such that $\bar{x}(i) \notin Q_\beta$.

Then i is odd and there exists $Q'_\beta \in L(0^{\beta\#_{\gamma+1}^1})[\#_\gamma^1]$ such that the position $p_1 = (Q'_\beta; \langle 0, \bar{x}(i) \rangle)$ is consistent with $s_{2\beta}^\gamma$. By Lemma 2.4.3, player II has a w.s. $s_1 \in L(\beta\#_{\gamma+1}^1(0))$ for $A(\bar{x}(i))$. Let $s(p) = s_1(p)$ for any position p which extends $\bar{x}(i)$.

Claim: The strategy s of player II is a w.s. for G_A .

Suppose x is a play of G_A consistent with s . By the definition of s , there exists a w.s. $s_0 \in L(\vec{Q})[\#_\gamma^1]$ for $A(\vec{Q})$ and $x(2i) = s_0(x(1); x(3); \dots; x(2i-1))$ whenever $\forall i' \leq 2i \bar{x}(i') \in Q_\beta = \bigcap_{1 \leq j \leq \beta} Q_j$. If $\forall i \bar{x}(i) \in Q_\beta$, then $x \notin A(\vec{Q})$ so that $x \notin A$.

Otherwise, there exists (odd) i such that $\bar{x}(i) \notin Q_\beta$ and $\forall j < i \bar{x}(j) \in Q_\beta$. By the definition of s , there exists a w.s. $s_1 \in L(\beta\#_{\gamma+1}^1(0))$ for $A(\bar{x}(i))$ such that $x(2k+1) = s_1(x(0); x(2); \dots; x(2k))$ whenever $2k+1 \geq i$. Therefore, since s_1 is a w.s. for II, $x \notin A(\bar{x}(i))$ so that $x \notin A$. Consequently, s is a w.s. in G_A of the player for whom $s_{2\beta}^\gamma$ is a w.s. ■

Definition 2.4. Let $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, \dots, B_1 \in \Pi_1^0$, $C_1, C_2, C_3, \dots, C_\beta \in \Sigma_1^0$, and $\langle A_\alpha \mid \alpha < \omega^2 \rangle$ strongly witness $A \in (\gamma * \Pi_1^0, \beta * \Sigma_1^0)^*$. Then we refer to the auxiliary game $G_{2\beta}^\gamma$ described in the Proof of Theorem 2.4 as *the $G_{2\beta}^\gamma$ auxiliary game determined by $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, \dots, B_1 \in \Pi_1^0$, $C_1, C_2, C_3, \dots, C_\beta \in \Sigma_1^0$, and $\langle A_\alpha \mid \alpha < \omega^2 \rangle$* .

Suppose $\vec{U} = \langle U_i \mid i < \beta \rangle$ and $\vec{u} = \langle u_i \mid i < \gamma \rangle$ respectively are a finite sequence of I-imposed subgames of G_A and a sequence of legal positions of

G_A . Then the $G_{2\beta}^\gamma(\vec{U}; \vec{u})$ auxiliary game determined by

$$B_\gamma, B_{\gamma-1}, B_{\gamma-2}, \dots, B_1, C_1, C_2, C_3, \dots, C_\beta, \langle A_\alpha \mid \alpha < \omega^2 \rangle, \text{ and } m \in \omega$$

is the game in which player I wins iff a position is reached at which II cannot make a (legal) move, which has exactly the same moves as $G_{2\beta}^\gamma$, and these moves are subject to the following conditions:

i.) $\{q \mid \exists k R_{C_i}(q, k)\}$ determines the Borel auxiliary moves Q_i and $\langle \hat{q}_i, q_i \rangle$ for $1 \leq i \leq \beta$.

ii.) The sequence $\langle (T_n^\delta; \langle \hat{t}_n^\delta, t_n^\delta \rangle) \mid \forall j < n \hat{t}_j^\delta = 0 \rangle$ of Borel auxiliary moves and the Π_1^0 set B_δ are related via R_{B_δ} .

iii.) If $1 \leq i < j \leq \beta$ and $\hat{q}_j = 1$, then $\hat{q}_i = 1$.

iv.) Each $\bar{x}(i) \in \bigcap_{j < \beta} U_j$ and each $\bar{x}(i)$ must be consistent with every u_j .

v.) Let $\vec{Q} = \langle Q_i \mid \hat{q}_i = 1 \rangle$, $\vec{q} = \langle q_i \mid \hat{q}_i = 0 \rangle$, and μ be least such that $\mu = \beta + 1$ or $\hat{q}_\mu = 0$. Then

$Q_i \in L(i \#_{\gamma+1}^1(\vec{U}))[\#_\gamma^1]$ for $i \geq \mu$ and $Q_i \in L((\mu - 1) \#_{\gamma+1}^1(\vec{U}))[\#_\gamma^1]$ for $i < \mu$.

vi.) $T_i^\delta \in L(\vec{U}, \vec{Q}, \vec{T}_{\delta-1})[\#_{\gamma+1-\delta}^1]$.

vii.) If $\hat{q}_\beta = 0$, let $k(0)$ be least such that $R_{C_\mu}(q_\mu, k(0))$; otherwise, $k(0)$ is undefined. The λ_i^δ 's are properly ordered with respect to

$$\langle A_\alpha \mid \alpha < \omega \cdot (k(\delta) + 1) \rangle \text{ using } \langle \omega_{i+1}^{L(\#_{\gamma+1-\delta}^1(\vec{U}, \vec{Q}, \vec{T}_\delta))} \mid i \leq k(\delta) \rangle.$$

viii.) If $\hat{t}_{k(\gamma)}^\gamma = 1$, the ξ_i 's are properly ordered with respect to

$$\langle A_\alpha \mid \alpha < \omega \cdot (k(\gamma) + 1) \rangle \text{ using } \langle \omega_{i+1}^{L(\#_1^1(\vec{U}, \vec{Q}, \vec{T}))} \mid i \leq k(\gamma) \rangle.$$

These conditions are analogous to the conditions for the moves of $G_{2\beta}^\gamma$.

The first three are conditions which the moves of $G_{2\beta}^\gamma$ also must satisfy. The others are derived by changing the conditions for the moves of $G_{2\beta}^\gamma$ so that we obtain conditions which are consistent with \vec{U} and \vec{u} . We refer to $G_{2\beta}^\gamma(\vec{U}; \vec{u})$ instead of the $G_{2\beta}^\gamma(\vec{U}; \vec{u})$ auxiliary game determined by

$$B_\gamma, B_{\gamma-1}, B_{\gamma-2}, \dots, B_1, C_1, C_2, C_3, \dots, C_\beta, \text{ and } \langle A_\alpha \mid \alpha < \omega^2 \rangle$$

whenever $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, \dots, B_1, C_1, C_2, C_3, \dots, C_\beta$, and $\langle A_\alpha \mid \alpha < \omega^2 \rangle$ are clear from the context. Analogous to Theorem 2.4, we have the following:

Corollary 2.4.1. Let $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, \dots, B_1, C_1, C_2, C_3, \dots, C_\beta, \langle A_\alpha \mid \alpha < \omega^2 \rangle, A, \vec{U}$, and \vec{u} be as in Definition 2.4. Let p be a legal position of a game G^* such that the moves of G^* following p constitute a play of $G_{2\beta}^\gamma(\vec{U}; \vec{u})$. Suppose \vec{U} has a wellordering which is definable in $L(\vec{U}), \beta\#_{\gamma+1}^1(\vec{U})$, and s^* is a w.s. for G^* such that $s^*|_{\ell_p} \in L(\beta\#_{\gamma+1}^1(\vec{U}))[\#_\gamma^1]$. If $L(\beta\#_{\gamma+1}^1(\vec{U}))[\#_\gamma^1] \models \text{“}r\#_\gamma^1\text{ exists for every real } r\text{”}$ then $s^*|_{\ell_p}$ can be integrated so as to obtain a w.s. $s^*|_{\ell_p} \in L(\beta\#_{\gamma+1}^1(\vec{U}))[\#_\gamma^1]$ for $A(\vec{U}; \vec{u})$ such that the following hold:

- i.) s_p is a w.s. of the player for whom s^* is a w.s.,
- ii.) If s^* is a w.s. for I, \hat{p} is a position consistent with s_p , and the moves in \hat{p} of player II are consistent with \vec{u} , then $\hat{p} \in \bigcap_{i < \beta} U_i$. Therefore, if s^* is a w.s. for I and x is a play consistent with s_p , then $x \in A(\vec{u})$.
- iii.) Let \hat{p} be a position consistent with s_p and with \vec{U} . If the moves in \hat{p} of the player for whom s_p is not a w.s. are consistent with \vec{u} , then \hat{p} is consistent with \vec{u} . ■

The proof of Corollaries 2.4.1 and 2.5.1 is similar to the proof of Theorems 2.4 and 2.5. However, we use a stronger induction hypothesis for the proof of the two corollaries. The statement of this stronger induction hypothesis follows Definition 2.5 (below).