Definition 0.12. $A \subseteq {}^{\omega}\omega$ is $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$ iff there exist B and $\langle A_\alpha | \alpha < \omega^2 \rangle$ which witness $A' \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$ and there exists $D \in \omega \cdot m - \Pi^1_1$ (for some $m \in \omega$) such that

$$x \in A \;\leftrightarrow\; (x \in A') \text{ or } (\forall n \neg B(x,n) \;\&\; x \in D).$$

In this case, we denote A by $B^*(\langle A_\alpha | \alpha < \omega^2 \rangle, D)$ and we say that B, $\langle A_\alpha | \alpha < \omega^2 \rangle$, and D *witness* that $A \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$. We say B, $\langle A_\alpha | \alpha < \omega^2 \rangle$, and D *strongly witness* $A \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$ if B and $\langle A_\alpha | \alpha < \omega^2 \rangle$ strongly witness $A' \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$.

We say that B, $\langle A_\alpha | \alpha < \omega^2 \rangle$, and $m \in \omega$ *witness* $A \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$ whenever B, $\langle A_\alpha | \alpha < \omega^2 \rangle$, and $A^*_{\omega \cdot m}$ witness $A \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$. If in addition $B = (R_1, R_2, R_3, ..., R_{\gamma+\beta})$, we say that $R_1, R_2, R_3, ..., R_{\gamma+\beta}$, $\langle A_\alpha | \alpha < \omega^2 \rangle$, and m *witness* $A \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$. Next we show that whenever we try to prove the determinacy of some $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$, we can assume wlog that each D_α (in the above definition) is A_α.

Lemma 0.13. If B, $\langle A'_\alpha | \alpha < \omega^2 \rangle$, and $D \in \omega \cdot (m+1) - \Pi^1_1$ *witness* that $A \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$, then there exists $\langle A_\alpha | \alpha < \omega^2 \rangle$ such that B, $\langle A_\alpha | \alpha < \omega^2 \rangle$, and m witness that $A \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$.

Proof: Let $\langle D_\alpha | \alpha \leq \omega \cdot (m+1) \rangle$ witness that $D \in (m+1) - \Pi^1_1$. Let $A_\alpha = \{x \in A'_\alpha | \exists n B(x,n)\} \cup \{x \in D_\alpha | \forall n \neg B(x,n)\}$. Since $B \in \Delta^1_1$, each $A_\alpha \in \Pi^1_1$. ∎

$A \subseteq {}^{\omega}\omega$ is defined to be in $\Delta(\omega^2 - \Pi^1_1)$ iff both A and its complement ${}^{\omega}\omega \setminus A$

are in $\omega^2 - \Pi^1_1$. $\Delta(\omega^2 - \Pi^1_1)$ properly contains $\bigcup_{\beta<\omega^2}(\beta - \Pi^1_1)$. Furthermore, it is not hard to show that for $\gamma \in \omega$ and $\beta \in \mathbf{N}$,

$$\textstyle\bigcup_{\alpha<\omega^2}(\alpha - \Pi^1_1) \subseteq (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^* \subset (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+ \subset \Delta(\omega^2 - \Pi^1_1)$$

and

$$(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^* \subset (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+ \subset (\hat{\gamma} * \Pi^0_1, \hat{\beta} * \Sigma^0_1)^* \subset (\hat{\gamma} * \Pi^0_1, \hat{\beta} * \Sigma^0_1)^*_+$$
$$\text{iff either } \gamma < \hat{\gamma} \text{ or } \gamma = \hat{\gamma} \text{ and } \beta < \hat{\beta}.$$

Also, $(\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_k)^* = (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$ and $(\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_k)^*_+ = (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$ whenever there exist $i_1 < i_2 < \cdots < i_\gamma$ such that $k = i_\gamma + \beta$, $\Gamma_i = \Pi^0_1$ for $i = i_1, i_2, ..., i_\gamma$, and otherwise $\Gamma_i = \Sigma^0_1$. In particular, $(\Sigma^0_1, \Pi^0_1)^* = (\Pi^0_1)^*$.

§0.3. Some Terminology for Games. In this paper, we show that for $\beta \in \omega$ and $\gamma \in \mathbf{N}$,

(i) the determinacy of $(\gamma * \Pi^0_1, \beta * \Sigma)^*$ follows from

$$L(\beta\#^1_{\gamma+1}(0))[\#^1_\gamma] \models \text{“}r^{\#^1_\gamma} \text{ exists for every real } r\text{,” and}$$

(ii) the determinacy of $(\gamma * \Pi^0_1, \beta * \Sigma)^*_+$ follows from the existence of $(\beta + 1)\#^1_{\gamma+1}(0)$.

In the proof of (i), we associate an auxiliary game $G^\gamma_{2\beta}$ with each B and $\langle A_\alpha | \alpha < \omega^2\rangle$ which witness a $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$ set. Similarly, in the proof of (ii), we associate an auxiliary game $G^\gamma_{2\beta+1}$ with each B, $\langle A_\alpha | \alpha < \omega^2\rangle$, and $m \in \omega$ which witness a $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$ set. The moves of $G^\gamma_{2\beta+1}$ are the same as the moves of $G^\gamma_{2\beta}$ except for the play of some ordinal auxiliary moves.

Whenever either $\hat{\gamma} < \gamma$ or $\hat{\gamma} = \gamma$ and $\hat{\beta} \leq \beta$, both the moves of $G^{\hat{\gamma}}_{2\hat{\beta}}$ and of $G^{\hat{\gamma}}_{2\hat{\beta}+1}$ are included in the game $G^{\gamma}_{2\beta+2}$.

We prove (i) and (ii) for several particular cases and then we refer to these proofs when showing (i) and (ii) hold in general. Since the auxiliary games G^{γ}_{β} are similar, we introduce in this section some terminology to avoid repetition. However, we first indicate that each G^{γ}_{β} has a w.s. with nice absoluteness properties.

For an arbitrary set X, define a topology on ${}^{\omega}X$ by letting B be a basic open set of ${}^{\omega}X$ iff there exist $x_0, x_1, x_2, ..., x_{n-1} \in X$ such that

$$B = \{f \in {}^{\omega}X | \forall i < n \ f(i) = x_i\}.$$

If A is an open set in ${}^{\omega}X$, then G_A is an *open game.*

By induction, we define the ordinals of each position of an open game: Suppose G is an open game. Then a position p in G has *ordinal* 0 if any play which is consistent with p is a win for I. If $p = (y(0), y(1), y(2), ..., y(2i))$ is a legal position in G and has odd length, then p has *ordinal* α iff for any move $y(2i+1)$, $(y(0), y(1), y(2), ..., y(2i+1))$ has ordinal less than or equal to α. If $p = (y(0), y(1), y(2), ..., y(2i-1))$ is a legal position in G and has even length, then p has ordinal α iff a move $y(2i)$ exists such that

$$(y(0), y(1), y(2), ..., y(2i)) \text{ has ordinal less than } \alpha.$$

We denote the set of all positions with ordinal α by $\mathbf{P}_{\alpha}(\mathbf{G})$, or by $\mathbf{P}_{\alpha}$ if G is clear from the context. We let $\mathbf{P}=\bigcup_{\alpha \in \mathrm{ON}} \mathbf{P}_{\alpha}$.

Each G^γ_β is an open game. Furthermore, open games are determined [GS] and have winning strategies with nice absoluteness properties. In the following lemma, we define such a w.s. for I when $\langle\ \rangle \in \mathbf{P}$ and for II when $\langle\ \rangle \notin \mathbf{P}$.

Lemma 0.14. [Folklore]. Let G be an open game (on an arbitrary set) and let E be the set of all legal positions p for the game G such that there is a play extending p that is won by II. Let $\prec$ be a wellordering of E. Then there is a w.s. s definable from $\prec$ in every inner model M of ZF such that $\prec \in M$ and $E \subseteq M$.

Proof: Let's refer to the moves of G as y(i) so that the play of G is:

$$G \quad \begin{array}{ccccccccc} \mathrm{I} & y(0) & & y(2) & & y(4) & & & \\ \mathrm{II} & & y(1) & & y(3) & & y(5) & \cdots & . \end{array}$$

Since G is open, the set P_α of positions with ordinal α is defined for all $\alpha \in \mathrm{ON}$, as is P. We shall see that s (defined below) will be a w.s. for I iff $\langle\ \rangle \in$P.

Let's first assume $\langle\ \rangle \in$P and describe the w.s. s (for I). Since $\langle\ \rangle \in$P, a least ordinal $\alpha(0)$ exists such that for some move y(0), we have $(y(0)) \in \mathrm{P}_{\alpha(0)}$; we let $s(\langle\ \rangle)$ be the move y(0) such that (y(0)) is the $\prec$–least position in $\mathrm{P}_{\alpha(0)}$. Now assume II plays y(1). Since $(y(0),y(1)) \in \bigcup_{\alpha \le \alpha(0)} \mathrm{P}_\alpha$, a least ordinal $\alpha(2) < \alpha(0)$ exists such that $(y(0),y(1),y(2)) \in \mathrm{P}_{\alpha(2)}$ for some y(2); we let s(y(1)) be the move y(2) such that (y(0);y(1);y(2)) is the $\prec$–least position in $\mathrm{P}_{\alpha(2)}$. In general, suppose $(y(0),y(1),y(2),\ldots,y(2i-1)) \in \bigcup_{\alpha \le \alpha(2i-2)} \mathrm{P}_\alpha$. Then a least ordinal $\alpha(2i) < \alpha(2i-2)$ exists such that (y(0),y(1),y(2),...,y(2i))$\in P_{\alpha(2i)}$ for some y(2i); we let s(y(1); y(3); y(5); . . . ; y(2i−1)) be the move y(2i) such that (y(0);y(1);y(2);...;y(2i)) is the $\prec$–least position in $P_{\alpha(2i)}$. Since a strictly decreasing sequence $\langle \alpha(2i) \mid i < \omega \rangle$ of ordinals can't exist, a position (y(0),y(1),y(2),...,y(2i)) must be reached at which no legal move y(2i+1) exists for II to play, and therefore II loses G.

Now assume $\langle\ \rangle \notin P$ and let's describe the w.s. s for II. If $\alpha < \beta$, then $P_\alpha \subseteq P_\beta$. Furthermore, P=$P_\alpha$ for some ordinal α—in fact, for the least α such that such that P_α=$P_{\alpha+1}$. Since $\langle\ \rangle \notin P_{\alpha+1}$, for any move y(0), (y(0))$\notin P_\alpha$. Since (y(0))$\notin$ P_α=$P_{\alpha+1}$, a move y(1) exists such that (y(0),y(1))$\notin P_\alpha$; let (y(0),y(1)) be the $\prec$–least such position and set s(y(0))=y(1). In general, suppose (y(0),y(1),y(2),...,y(2i))$\notin P_{\alpha+1}$; then there is a move y(2i+1) such that (y(0),y(1),y(2),...,y(2i+1)) is the $\prec$–least position not in P_α and set s(y(0),y(1),y(2),...,y(2i))=y(2i+1). If y were a win for I, then some position (y(0),y(1),y(2),...,y(i)) would be in P_0; therefore, since (y(0),y(1),y(2),...,y(i))$\notin$P for all i, s is a w.s. for II.

In both cases, s depends only on $\prec$ and P. Therefore, s is definable from $\prec$ in every inner model M of ZF such that $\prec \in M$ and $E \subseteq M$. ■

As mentioned above, the auxiliary games G^γ_β (to be later defined) are open. Therefore, for each β and γ, Lemma 0.14 will provide us with a w.s. s^γ_β for G^γ_β, and we will refer to s^γ_β as the *canonical w.s. for* G^γ_β.

The auxilary games G^{γ}_{β} include ordinal auxiliary moves ξ_i for $i \in \omega$. ξ_i is played by I if i is even, is played by II if i is odd, and is played before ξ_j if $i < j$. There are certain restrictions placed on each ξ_i and we think of each ξ_i as some particular ξ^{α}_j. Fix $m \in \omega$. We now define for $\alpha < \omega{\cdot}m$, $\pi_{\alpha}{:}\omega \to \omega$ with certain properties and set $\xi^{\alpha}_j{=}\xi_{\pi_{\alpha}(j)}$.

Fix $m \in \omega$ and let $\pi{:}\omega \to \omega{\cdot}m$ be recursive such that $\pi(2n)$ is even and $\pi(2n+1)$ is odd for $n \in \omega$, $\pi^{-1}(\alpha)$ is infinite for each $\alpha < \omega{\cdot}m$ and if $i < j < \omega$, then for each $n \in \omega$, the least element of $\pi^{-1}(\omega \cdot n + i)$ is less than the least element of $\pi^{-1}(\omega{\cdot}n + j)$. Let $\pi_{\alpha} : \omega \to \pi^{-1}(\alpha)$ be the bijection defined by "$\pi_{\alpha}(i)$ is the least element of $\pi^{-1}(\alpha) \setminus \{\pi_{\alpha}(\mathrm{j}) | \mathrm{j} < \mathrm{i}\}$." Then

$\{\pi_{\alpha}(i) | \alpha < \omega{\cdot}m,\ \alpha\ \mathit{is\ even},\ i \in \omega\}$ is the set of even integers,

$\{\pi_{\alpha}(i) | \alpha < \omega{\cdot}m,\ \alpha\ \mathit{is\ odd},\ i \in \omega\}$ is the set of odd integers,

$\pi_{\omega \cdot n+k}(0) < \pi_{\omega \cdot n+k+1}(0)$ for $n < m$ and $k \in \omega$,

$\pi_{\alpha}(i) \neq \pi_{\beta}(j)$ if $\alpha \neq \beta$ or $i \neq j$, and

$\pi_{\alpha}(i) < \pi_{\alpha}(j)$ if $i < j$.

Finally, let ξ^{α}_j abbreviate $\xi_{\pi_{\alpha}(j)}$.

Notice that ξ^{α}_j is played before ξ^{α}_{j+1} and $\xi^{\omega \cdot n+k}_0$ is played before $\xi^{\omega \cdot n+k+1}_0$ for $n < m$ and $k \in \omega$. Furthermore, ξ^{α}_j is played by I if α is even and by II if α is odd.

The ordinal auxiliary moves ξ_i will provide the players with a way to show that a particular real $x \in {}^{\omega}\omega$ is in a given Π^1_1 set iff a certain ordering, determined by x, is a wellordering. The next lemma provides us with such orderings.

Lemma 0.15. Kleene [Kl]. Let β be a recursive ordinal and $\langle A_\alpha | \alpha < \beta \rangle$ witness $A \in \beta - \Pi^1_1$ (so that there exists a recursive wellordering of a subset E of ω with order type β such that

$$\{(k, x) \in E \times {}^\omega\omega | x \in A_{|k|}\} \in \Pi^1_1).$$

Then there exists a recursive function F with domain

$E \times \{\bar{x}(i) | x \in {}^\omega\omega \text{ and } i \in \omega\}$ such that

1.) $F(n, \bar{x}(i))$ is a linear ordering of $0, 1, 2, 3, ..., i$ with largest element 0,

2.) $F(n, \bar{x}(i))$ is a subordering of $F(n, \bar{x}(j))$ if $i \leq j$, and

3.) $x \in A_{|n|}$ iff $\bigcup_{i \in \omega} F(n, \bar{x}(i))$ is a wellordering.

In the above situation, let $F_{|n|}$ be the function with domain

$\{\bar{x}(i) | i \in \omega \text{ and } x \in {}^\omega\omega\}$ such that $F_{|n|}(j) = F(n, j)$ so that

$$x \in A_\alpha \text{ iff } \bigcup_{i \in \omega} F_\alpha(\bar{x}(i)) \text{ is a wellordering.} \qquad \blacksquare$$

The moves ξ^α_j are used to determine if x is in some fixed $A_\alpha \in \Pi^1_1$. To make use of the ξ^α_j, we place certain restrictions on the ξ^α_j. The following terminology is used to describe some of these restrictions.

Definition 0.16. Let G^* be a game with integer moves $x(j)$ for $j \in \omega$ and with ordinal moves $\xi_0, \xi_1, \xi_2, ..., \xi_{i-1}$ such that

i.) $x(j)$ is played before $x(k)$ if $j < k$

and

ii.) $x(j)$ and ξ_j are each played before ξ_k if $j < k < i$.

Let p be a position in G^* which includes the ordinal moves $\xi_0, \xi_1, \xi_2, ..., \xi_{i-1}$. Let $\langle A_\alpha \mid \alpha \leq \omega \cdot m\rangle$ witness that some set is $\omega \cdot m - \Pi^1_1$, and let F be a recursive function as in Lemma 0.15 for the case $\beta = \omega \cdot m$ so that

$$x \in A_\alpha \text{ iff } \bigcup_{j\in\omega} F_\alpha(\bar{x}(j)) \text{ is a wellordering.}$$

Recall π and π_α which are defined just before Lemma 0.15. Whenever $\pi_\alpha(j) < i$, let $\xi^\alpha_j = \xi_{\pi_\alpha(\mathrm{j})}$. Notice that whenever $\pi_\alpha(j) < i$, $j \leq \pi_\alpha(j) < i$ and $\pi_\alpha(0) < \pi_\alpha(1) < \pi_\alpha(2) < \cdots < \pi_\alpha(j)$ so that p includes the moves $x(0), x(1), x(2), ..., x(j-1)$ and $\xi^\alpha_0, \xi^\alpha_1, \xi^\alpha_2, ..., \xi^\alpha_j$. We say that *($\xi_0, \xi_1, \xi_2, ..., \xi_{i-1}$) is properly ordered with respect to*

$$(x(0), x(1), x(2), ..., x(i-1)) \textit{ (or just } \bar{x}(i)\textit{) and } \langle A_\alpha \mid \alpha < \omega{\cdot}m\rangle$$

in G^* using $\langle \kappa_{n+1} \mid n < m\rangle$ iff whenever $\pi_\alpha(\mathrm{j}) < \mathrm{i}$, we have

i.) $\xi^\alpha_{\mathrm{j}} \in \kappa_{\mathrm{n}+1}$ if $\alpha = \omega{\cdot}\mathrm{n}{+}\mathrm{k}$ for some $k \in \omega$ (and $n < m$), and

ii.) the map from $F_\alpha(\bar{x}(j+1))$ into the ordinals defined by $k \mapsto \xi^\alpha_k$ is order preserving.

We say that *($\xi_0, \xi_1, \xi_2, ..., \xi_{i-1}$) is properly ordered with respect to $\bar{x}(i)$ using $\langle \kappa_{n+1} \mid n < m\rangle$* whenever G^* and $\langle A_\alpha \mid \alpha < \omega \cdot m\rangle$ are clear from the context and $(\xi_0, \xi_1, \xi_2, ..., \xi_{i-1})$ is properly ordered with respect to $\bar{x}(i)$ and $\langle A_\alpha \mid \alpha < \omega{\cdot}m\rangle$ in G^* using $\langle \kappa_{i+1} \mid i < m\rangle$. In this paper, G^* is always some G^γ_β and $\langle A_\alpha \mid \alpha < \omega \cdot m\rangle$ is clear from the context. The auxiliary games G^γ_β include integer moves $x(i) \in \omega$ and ordinal auxiliary moves ξ_j such that we require the players to only play so that $(\xi_0, \xi_1, \xi_2, ..., \xi_{i-1})$ is properly ordered with respect to $\bar{x}(i)$ using some particular sequence $\langle \kappa_{i+1} \mid i < m \rangle$. Theorem 0.8 [Du1] provides us with a concrete example of ordinal auxiliary moves being required to be properly ordered: In Theorem 0.8 [Du1], the players are required to play ξ_j such that $(\xi_0, \xi_1, \xi_2, ..., \xi_{i-1})$ is properly ordered with respect to $\bar{x}(i)$ using $\langle \omega_{i+1} \mid i < m \rangle$. In such games, $\bar{x}(i)$ is usually clear from the context so that we just say that the auxiliary ordinal moves ξ_j must be properly ordered using $\langle \kappa_{i+1} \mid i < m \rangle$.

Each game G^γ_β is an auxiliary game for some G_A, and player I plays in G^γ_β a set T of positions in G_A such that if $\bar{x}(2i+1) \in T$, then $\bar{x}(2i+2) \in T$ for any $x(2i+1) \in \omega$. We say that T is an *I-imposed subgame* of the game G_A if T is a set of positions of G_A such that $p \in T$ whenever p is a position of even length, say of length 2i+2, which extends some position $p' \in T$ of length 2i+1. Thus, an I-imposed subgame of G_A is a set of positions which restricts I's moves but does not restrict II's moves.

Let G_A be a game with integer moves $x(i)$. Let $\vec{U} = \langle U_i | i < n \rangle$ be a sequence of subsets of $\omega^{<\omega}$, and let $\vec{u} = \langle u_i | i < m \rangle$ be a sequence of elements from $\omega^{<\omega}$. (Often $\vec{U}$ is a sequence of I-imposed subgames of G_A, and $\vec{u}$ is a sequence of legal positions of G_A with odd length.) Define $G_A(\vec{U}; \vec{u})$ to be the game whose moves are subject to the same conditions as those of G_A except that the players may only play $x(i-1)$ such that each $\bar{x}(i) \in \bigcap_{j<n} U_j$ and

each $\bar{x}(i)$ is compatible with each u_j for $j < m$.

Each G^γ_β has designated integer moves $x(i)$ (as well as some other moves). In this paper, we also define $G^\gamma_\beta(\vec{U};\vec{u})$. $G^\gamma_\beta(\vec{U};\vec{u})$ has exactly the same moves as G^γ_β. Furthermore, the moves of $G^\gamma_\beta(\vec{U};\vec{u})$ and G^γ_β are subject to many of the same conditions. However, in $G^\gamma_\beta(\vec{U};\vec{u})$, the players may only play $x(i-1)$ such that each $\bar{x}(i) \in \bigcap_{j<n} U_j$ and each $\bar{x}(i)$ is compatible with each u_j for $j < m$. Also, if $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, ..., \lambda_{i-1})$ is a sequence of ordinal moves in some G^n_k which must be properly ordered with respect to $\bar{x}(i)$ using $\langle \omega_{i+1}^{L(\beta \#^1_{\gamma+1}(\vec{T}))} | i \leq m \rangle$, then in $G^n_k(\vec{U};\vec{u})$, $(\lambda_0, \lambda_1, \lambda_2, ..., \lambda_{i-1})$ must be properly ordered with respect to $\bar{x}(i)$ using $\langle \omega_{i+1}^{L(\beta \#^1_{\gamma+1}(\vec{U},\vec{T}))} | i \leq m \rangle$. We explicitly state the conditions on the moves of G^γ_β later.

If G^* is either some G^γ_β or some G_A with $A \subseteq \omega^{<\omega}$, we sometimes write $G^*(U_0, U_1, U_2, ..., U_{n-1}; u_0, u_1, u_2, ..., u_{m-1})$ for $G^*(\vec{U};\vec{u})$. Also, if p is a legal position of a game G, then G_p is the game in which both players are required to play so that all positions are consistent with p, and if neither player loses by failing to meet this requirement, then the winning conditions for G_p are exactly the same as for G. Therefore, if $A \subseteq \omega^{<\omega}$ and $p \in \omega^{<\omega}$, then $(G_A)_p$ and $G_A(p)$ are the same game.

The games G^γ_β include the play of several Borel auxiliary moves T and $\langle \hat{t}, t \rangle$, each of which must satisfy certain conditions. These conditions are enumerated in the following:

Definition 0.17. Let $R \subseteq \omega$ and let G be a game with only integer moves $x(i)$ for $i \in \omega$. Let $\vec{U} = \langle U_i \mid i < \eta \rangle$ and $\vec{u} = \langle u_i \mid i < \theta \rangle$ respectively be a finite sequence of I-imposed subgames of G and a sequence of legal positions of G. Let G^* be a game with designated integer moves $x(i)$ for $i \in \omega$, Borel auxiliary moves T and $\langle \hat{t}, t \rangle$, and maybe some other moves. Assume that in both G and G^* the play of each $x(i-1)$ precedes the play of $x(i)$. (For our purposes, G^* is some $G^{\gamma}_{\beta}(\vec{U}; \vec{u})$.) *$T$ and $\langle \hat{t}, t \rangle$ are Borel auxiliary moves of G^* with respect to $G(\vec{U}; \vec{u})$* if the following conditions hold:

(1) Player I plays T and the next move of II is $\langle \hat{t}, t \rangle$.

(2) If $\langle x(i) \mid i < k \rangle$and $\langle Q_i; \langle \hat{q}_i, q_i \rangle \mid i < n \rangle$ respectively are the sequences of integer moves $x(i)$ and Borel auxiliary moves which precede the play of T in G^*, then T is an I-imposed subgame of $G(\vec{U}, \langle Q_i \mid i < n \rangle; \vec{u}, \langle q_i \mid i < n \rangle, \bar{x}(k))$.

(3) $\langle \hat{t}, t \rangle \in \{\langle 1, - \rangle, \langle 0, q \rangle | q \in T, \ \textit{and } q \textit{ has odd length}\}$.

(4) If II plays $\hat{t} = 0$, then both players may only play integer moves $x(i-1)$ which are consistent with t i.e. $\bar{x}(i)$ and t must be compatible.

We say that *R and $G(\vec{U}; \vec{u})$ determines the Borel auxiliary moves T and $\langle \vec{t}, t \rangle$ of G^** if the following hold in addition to (1) through (4):

(5) If $\hat{t} = 0$, then $R(t)$.

(6) If integer moves $x(0), x(1), x(2), ..., x(i-1)$ precede the play of the move T and $R(\bar{x}(i))$, then II must play so that $\hat{t} = 0$.

(7) If Borel auxiliary moves Q and $\langle \hat{q}, q \rangle = \langle 0, \bar{x}(2i-1) \rangle$ precede the play

of the move T and $\exists j < 2i\ R(\bar{x}(j))$, then II must play so that $\hat{t} = 0$.

(8) If II plays $\hat{t} = 1$, then only integer moves $x(i-1)$ such that $\neg R(\bar{x}(i))$ may be played in G^*.

We say that R determines the Borel auxiliary moves T and $\langle \hat{t}, t\rangle$ of G^* whenever R and G determine T and $\langle \hat{t}, t\rangle$ and G is understood from the context. In the auxiliary games G^γ_β, we restrict I to play each Borel auxiliary move T so that T belongs to a certain model (which is later specified).

Any play of the auxiliary game G' of Theorem 1.5 [Du2], which includes Borel auxiliary moves T_i and $\langle \hat{t}_i, t_i\rangle$=$\langle 0, t_i\rangle$ for $i < n$, also includes the Borel auxiliary moves T_n and $\langle \hat{t}_n, t_n\rangle$ determined by $\{m \in \omega | R_B(m, n)\}$ and G_A. Furthermore, for any play of G' which includes $\langle \hat{t}_n, t_n\rangle$, $B(x, n)$ iff $\hat{t}_n = 1$. The auxiliary games G^γ_β also satisfy the above conditions and in each G^γ_β, we have a similar relation between a Π^0_1 set B and a sequence

$$\langle (T_n; \langle \hat{t}_n, t_n\rangle) | \forall j < n\ \hat{t}_j = 0\rangle$$

of Borel auxiliary moves. Therefore, we make the following definition:

Definition 0.18. Let $B \subseteq ({}^\omega\omega) \times \omega$ and $R \in \Sigma^0_1$ satisfy

$$B(x, n) \leftrightarrow \forall k R(\bar{x}(k), n).$$

The sequence $\langle (T_n; \langle \hat{t}_n, t_n\rangle) | \forall j < n\ \hat{t}_j = 0\rangle$ of Borel auxiliary moves of G^* and the Π^0_1 set B *are related* via R if the following two conditions hold:

(1) Every play of G^* includes the moves T_i and $\langle \hat{t}_i, t_i\rangle = \langle 0, t_i\rangle$ for all $i < n$ iff G^* includes Borel auxiliary moves T_n and $\langle \hat{t}_n, t_n\rangle$.

(2) If G^* includes Borel auxilary moves T_i, $\langle \hat{t}_i, t_i\rangle$, T_j, and $\langle \hat{t}_j, t_j\rangle$ and $i \leq j$, then T_i and $\langle \hat{t}_i, t_i\rangle$ are Borel auxiliary moves of G^* determined by $\{m \in \omega | R(m,i)\}$ and the move $\langle \hat{t}_j, t_j\rangle$ does not precede the move T_i.

If the sequence $\langle (T_n; \langle \hat{t}_n, t_n\rangle) | \forall j < n \ \hat{t}_j = 0\rangle$ of Borel auxiliary moves of G^* and the Π^0_1 set B *are related* via R, we say G^* contains the sequence $\langle (T_n; \langle \hat{t}_n, t_n\rangle) | \forall j < n \ \hat{t}_j = 0\rangle$ which is related to the Π^0_1 set B via R.

§0.4. The Auxiliary Games G^0_0, G^1_0, and G^1_1. We briefly review the definitions of the auxiliary games $\bar{G}$ of Theorem 0.8 [Du1] and G' and G^* of Theorems 1.5 and 1.6 [Du2]. We respectively refer to $\bar{G}$, G', and G^* as G^0_0, G^1_0 and G^1_1 here.

§0.4.1. The Auxiliary Game G^0_0. Martin showed that if

$$A \in \bigcup_{\alpha < \omega^2} (\alpha - \Pi^1_1),$$

$\vec{U}$ is a sequence of I-imposed subgames of G_A, $\vec{u}$ is a sequence of legal positions for G_A, and $\#_1(\vec{U})$ exists, then the game $A(\vec{U}; \vec{u})$ is determined. In the proof of this theorem, we obtain a w.s. for $A(\vec{U}; \vec{u})$ by integrating a w.s. for the auxiliary game $G^0_0(\vec{U}; \vec{u})$, which we now define.

Definition 0.19. Let $m \in \mathbf{N}$ and let $\langle A_\alpha \mid \alpha \leq \omega \cdot m\rangle$ witness $A \in \omega \cdot m - \Pi^1_1$. Let $\vec{U} = \langle U_i | i < \eta\rangle$ and $\vec{u} = \langle u_i | i < \theta\rangle$ respectively be a finite sequence of I-imposed subgames of G_A and a sequence of legal positions of G_A. We describe the $G^0_0(\vec{U}; \vec{u})$ *game associated with* $\langle A_\alpha | \alpha \leq \omega \cdot m\rangle$ and refer to this game as $G^0_0(\vec{U}; \vec{u})$. In $G^0_0(\vec{U}; \vec{u})$, an ordinal auxiliary move ξ_i is played with each integer move $x(i)$:

$$G_0^0(\vec{U};\vec{u}) \quad \begin{array}{lllllllll} \mathrm{I} & x(0),\xi_0 & & x(2),\xi_2 & & x(4),\xi_4 & & \cdots & x(2n),\xi_{2n} & & \\ \mathrm{II} & & x(1),\xi_1 & & x(3),\xi_3 & & x(5),\xi_5 & & & x(2n{+}1),\xi_{2n+1} & \cdots . \end{array}$$

Each $\bar{x}(i) \in \bigcap_{i<\eta} U_i$ and each $\bar{x}(i)$ must be consistent with every u_i. The ξ_i are properly ordered with respect to $\langle A_\alpha | \alpha < \omega{\cdot}m \rangle$ using $\langle \omega_{i+1}^{L(\#_1(\vec{U}))} | i < m \rangle$. Player I wins G_0^0 iff a position is reached at which II cannot make a (legal) move. If $\vec{U}$ and $\vec{u}$ are both the empty sequence, we write G_0^0 for $G_0^0(\vec{U};\vec{u})$.

In Theorem 1.8 [Du1], we give the proof of Martin's Theorem: If $0^\#$ exists and $\langle D_\alpha | \alpha \leq \omega \cdot m \rangle$ witness D is $\omega \cdot m - \Pi_1^1$, then G_D has a w.s. $s \in L(\#_1(0))$. In that proof, we obtain the w.s. $s \in L(0^\#)$ for G_D by integrating the w.s. s_0^0 (for the G_0^0 game associated with $\langle D_\alpha | \alpha \leq \omega \cdot m \rangle$) with respect to the ordinal auxiliary moves ξ_i. (In that proof, s_0^0 and G_0^0 are respectively referred to as $\bar{s}$ and $\bar{G}$.) For fixed $(\vec{U};\vec{u})$, the w.s. s_0^0 for the $G_0^0(\vec{U};\vec{u})$ game associated with $\langle D_\alpha | \alpha \leq \omega \cdot m \rangle$ can similarly be integrated with respect to the ξ_i so as to obtain a w.s. $s \in L(\#_1(\vec{U}))$ for $D(\vec{U};\vec{u})$. Moreover, we similarly have the following:

Theorem 0.20. Let m, $\langle A_\alpha \mid \alpha \leq \omega \cdot m \rangle$, A, $\vec{U}$, and $\vec{u}$ be as in Definition 0.19. Let p be a legal position of a game G^* such that the moves of G^* following p constitute a play of $G_0^0(\vec{U};\vec{u})$. Suppose s^* is a w.s. for G^* such that $s^*|\ell_p \in L(\vec{U})$. Then $s^*|\ell_p$ can be integrated so as to obtain a w.s. $s_p \in L(\#_1(\vec{U}))$ for $A(\vec{U};\vec{u})$ such that the following hold:

(i) s_p is a w.s. of the player for whom s^* is a w.s.,

(ii) If s^* is a w.s. for I, $\hat{p}$ is a position consistent with s_p, and the moves in $\hat{p}$ of player II are consistent with $\vec{u}$, then $\hat{p} \in \bigcap_{i<\eta} U_i$. Therefore, if s^* is a w.s. for I and x is a play consistent with s_p, then $x \in A(\vec{u})$.

(iii) Let $\hat{p}$ be a position consistent with s_p and with $\vec{U}$. If the moves in $\hat{p}$ of the player for whom s_p is not a w.s. are consistent with $\vec{u}$, then $\hat{p}$ is consistent with $\vec{u}$.

Proof: In Theorem 1.8 [Du1], we integrate the w.s. s_0^0 for G_0^0 with respect to the ordinal auxiliary moves ξ_i. In G_p^*, similarly integrate $s^*|\ell_p$ with respect to the ξ_i. ■

§**0.4.2. The Game G_0^1.** Assume $L[\#_1] \models$ "every real has indiscernibles;" otherwise, G_0^1 is undefined. Let $B \in \Pi_1^0$ and $\langle A_\alpha \mid \alpha < \omega^2 \rangle$ strongly witness $A \in (\Pi_1^0)^*$, and let $R \in \Delta_1^0$ be such that $B(x,n) \leftrightarrow \forall k R(\bar{x}(k), n)$ and such that

$$\text{if } \neg\mathrm{R}(\bar{\mathrm{x}}(\mathrm{k}),\mathrm{n})\ \&\ \forall \mathrm{j} < \mathrm{k}\mathrm{R}(\bar{\mathrm{x}}(\mathrm{j}),\mathrm{n}),\text{ then k is odd.}$$

We describe the G_0^1 *game associated with B and $\langle A_\alpha | \alpha < \omega^2 \rangle$*. G_0^1 contains a sequence $\langle (T_n; \langle \hat{t}_n, t_n \rangle) | \forall j < n\ \hat{t}_j = 0 \rangle$ of Borel auxiliary moves which is related to B via R. If all $\hat{t}_i = 0$, then the play of G_0^1 is

$$\begin{array}{lccccccccc} \mathrm{I} & \mathrm{T}_0 & & \mathrm{x}(0) & & \mathrm{T}_1 & & \mathrm{x}(2) & & \\ \mathrm{II} & & \langle 0,\mathrm{t}_0 \rangle & & \mathrm{x}(1) & & \langle 0,\mathrm{t}_1 \rangle & & \mathrm{x}(3) & \cdots . \end{array}$$

Notice in this case that no ordinal auxiliary moves are played. If $\hat{t}_n = 1$ is played, ordinal auxiliary moves ξ_i are played with the integer moves $x(2n+i)$; in this case, the play of G_0^1 is

$$\begin{array}{lcccccccc} \text{I} & T_0 & & x(0) & & T_1 & & x(2) & \\ \text{II} & & \langle 0,t_0\rangle & & x(1) & & \langle 0,t_1\rangle & & x(3) \end{array}\cdots.$$

$$\cdots\begin{array}{cccccccccc} T_{n-1} & & x(2n-2) & & T_n & & x(2n),\xi_0 & & x(2n+2),\xi_2 & \\ & \langle 0,t_{n-1}\rangle & & x(2n-1) & & \langle 1,-\rangle & & x(2n+1),\xi_1 & & x(2n+3),\xi_3 \end{array}\cdots.$$

The ξ_i are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (n+1)\rangle$ using $\langle \omega_{i+1}^{L(\#_1(T_n))} | i \leq n\rangle$.

DuBose [Du2] uses the game G_0^1 to show the determinacy of $(\Pi_1^0)^*$ follows from $L[\#_1] \models$"every real has a sharp."

§**0.4.3. The Game G_1^1.** Assume $L[\#_1]$ has indiscernibles; otherwise, G_1^1 is undefined. Let $B \in \Pi_1^0$, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, and $D \in \omega \cdot (m+1) - \Pi_1^1$ strongly witness $A \in (\Pi_1^0)_+^*$, and let $R \in \Delta_1^0$ be such that $B(x,n) \leftrightarrow \forall k R(\bar{x}(k), n)$ and such that

$$\text{if } \neg R(\bar{x}(k), n) \;\&\; \forall j < k R(\bar{x}(j), n), \text{ then } k \text{ is odd.}$$

We describe the G_1^1 *game associated with* B, $\langle A_\alpha | \alpha < \omega^2\rangle$, *and* D. G_1^1 contains all the moves of G_0^1. Whenever II plays $\hat{t}_i = 0$, ordinal auxiliary moves λ_i and λ_{i+1} are respectively played with the integer moves $x(i)$ and $x(i+1)$. The λ_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (m+1)\rangle$ using $\langle \omega_{i+1}^{L(\#_2^1(0))} | i \leq m\rangle$. If all $\hat{t}_i = 0$, then the play of G_1^1 is

$$\begin{array}{lccccccccccc} \text{I} & T_0 & & x(0),\lambda_0 & & T_1 & & x(2),\lambda_2 & & T_2 & & x(4),\lambda_4 & \\ \text{II} & & \langle 0,t_0\rangle & & x(1),\lambda_1 & & \langle 0,t_1\rangle & & x(3),\lambda_3 & & \langle 0,t_2\rangle & & x(5),\lambda_5 \end{array}\cdots\,.$$

If some $\hat{t}_n = 1$ is played, then the play of G_1^0 is

$$\begin{array}{lcccccccc} \text{I} & T_0 & & x(0),\lambda_0 & & T_1 & & x(2),\lambda_2 & \\ \text{II} & & \langle 0,t_0\rangle & & x(1),\lambda_1 & & \langle 0,t_1\rangle & & x(3),\lambda_3 \end{array}\cdots$$

$$\cdots\begin{array}{cccccccccc} T_{n-1} & & x(2n-2),\lambda_{2n-2} & & T_n & & x(2n),\xi_0 & & x(2n+2),\xi_2 & \\ & \langle 0,t_{n-1}\rangle & & x(2n-1),\lambda_{2n-1} & & \langle 1,-\rangle & & x(2n+1),\xi_1 & & x(2n+3),\xi_3 \end{array}\cdots\,.$$

Analogous to G_0^1, G_1^1 contains the sequence $\langle (T_n; \langle \hat{t}_n, t_n\rangle) | \forall j < n\ \hat{t}_j = 0\rangle$ of

Borel auxiliary moves which is related to B via R, and the ξ_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (n+1) \rangle$ using $\langle \omega_{i+1}^{L(\#_1(T_n))} | i \leq n \rangle$.

DuBose [Du2] uses the game G_1^1 to show the determinacy of $(\Pi_1^0)_+^*$ follows from the existence of $0^{\#_2^1}$ (i.e. from $L[\#_1]$ has indiscernibles).