§1. **The Existence of $0^{\beta\#^1_2}$ Implies Determinacy.**

In this section, we show the following two theorems:

Theorem 1.6. If $L(0^{\beta\#^1_2})[\#^1_1] \models$ "every real has a sharp,"

then $\mathrm{Det}(\Pi^0_1, \beta * \Sigma^0_1)^*$.

Theorem 1.7. If $L(0^{\beta\#^1_2})[\#^1_1]$ has indiscernibles, then $\mathrm{Det}(\Pi^0_1, \beta * \Sigma^0_1)^*$.

However, we first show some special cases (mentioned below) of the above results. Figure 5 illustrates the theorems of this section.

We also generalize Theorem 1.7 by showing the following:

Corollary 1.7.1. If $A \in (\Pi^0_1, \beta * \Sigma^0_1)^*_+$, $\vec{U}$ is a finite sequence of I-imposed subgames of G_A, $\vec{u}$ is a sequence of legal positions of G_A with odd length, and $\beta\#^1_2(\vec{U})$ exists, then $G_A(\vec{U};\vec{u})$ has a w.s. in $L(\beta\#^1_2(\vec{U}))$.

We prove the theorems of this section by integrating winning strategies for auxiliary games G^1_β. In this section, we write G^β for G^1_β. Therefore, we resepectively write G^0 and G^1 for the games G^1_0 and G^1_1 described in §0.4. However, in Sections Zero and Two, we only use the notation G^1_β since in those sections we consider G^γ_β for $\gamma \neq 1$.

The proof of Theorems 1.6, 1.7, and Corollary 1.7.1 is by induction on β. We prove Theorems 1.6, 1.7, and Corollary 1.7.1 for $\beta = 1, 2$ before proving the general result. In Theorem 1.1, we also show Corollary 1.7.1 for $\beta = 0$. In Theorem 1.0, we show that if $A \in (\Pi^0_1)^*$, $\vec{U}$ is a finite sequence of I-imposed subgames of G_A, $\vec{u}$ is a sequence of legal positions of G_A with odd length, and

$L(\#^1_2(\vec{U}))[\#_1] \models$ "every real has indiscernibles," then $G_A(\vec{U};\vec{u})$ has a w.s. in $L(\#^1_2(\vec{U}))[\#_1]$. We will need Theorems 1.0 and 1.1 in the proof of Theorems 1.6, 1.7, and Corollary 1.7.1. However, the reader may (initially) want to skip the proofs of Theorems 1.0 and 1.1.

In the proof of Theorem 1.0, we integrate a w.s. for an auxiliary game $G^1_0(\vec{U};\vec{u})$. $G^1_0(\vec{U};\vec{u})$ is a generalization of the game G^1_0 described in §0.4.2, which we now define.

Definition 1.0. Let $B \in \Pi^0_1$ and $\langle A_\alpha \mid \alpha < \omega^2\rangle$ strongly witness $A \in (\Pi^0_1)^*$. Let $\vec{U} = \langle U_i | i < \eta\rangle$ and $\vec{u} = \langle u_i | i < \theta\rangle$ respectively be a finite sequence of I-imposed subgames of G_A and a sequence of legal positions of G_A. Then the $G^0(\vec{U};\vec{u})$ *auxiliary game determined by* B *and* $\langle A_\alpha | \alpha < \omega^2\rangle$ is the game which has exactly the same moves as G^0 and these moves are subject to the following conditions:

i.) Each $\bar{x}(i) \in \bigcap_{i<\eta} U_i$ and each $\bar{x}(i)$ must be consistent with every u_i.

ii.) $T_i \in L(\vec{U})[\#_1]$.

iii.) If $\hat{t}_n = 1$, the ξ_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega\cdot(n+1)\rangle$ using $\langle \omega_{i+1}^{L(\#^1_1(\vec{U},T_n))} | i \leq n\rangle$.

iv.) The sequence $\langle (T_n; \langle \hat{t}_n, t_n\rangle) | \forall j < n\ \hat{t}_j = 0\rangle$ of Borel auxiliary moves and the Π^0_1 set B are related via R_B.

Player I wins $G^0(\vec{U};\vec{u})$ iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move.

These conditions are analogous to the conditions for the moves of G^0. They are derived by changing the conditions for the moves of G^0 so that they are consistent with $\vec{U}$ and $\vec{u}$. We refer to $G^0(\vec{U};\vec{u})$ instead of the $G^0(\vec{U};\vec{u})$ auxiliary game determined by B and $\langle A_\alpha \mid \alpha < \omega^2\rangle$ whenever B, C, and $\langle A_\alpha \mid \alpha < \omega^2\rangle$ are clear from the context. In [Du2], we obtain the following by integrating a w.s. for G^0: If $L[\#_1] \models$ "every real has indiscernibles," then every $(\Pi^0_1)^*$ game is determined. Similarly, we prove the following:

Theorem 1.0. Let B, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, A, $\vec{U}$, and $\vec{u}$ be as in Definition 1.0. Let $\hat{p}$ be a legal position of a game G^* such that the moves of G^* following $\hat{p}$ constitute a play of $G^0(\vec{U};\vec{u})$. Suppose $\vec{U}$ has a definable wellordering in $L(\vec{U})$ and s^* is a w.s. for G^* such that $s^*|\ell_{\hat{p}} \in L(\vec{U})[\#_1]$. If $L(\vec{U})[\#_1] \models$ "every real has a sharp," then $s^*|\ell_{\hat{p}}$ can be integrated so as to obtain a w.s. $s_{\hat{p}} \in L(\vec{U})[\#_1]$ for $A(\vec{U};\vec{u})$ such that the following hold:

(i) $s_{\hat{p}}$ is a w.s. of the player for whom s^* is a w.s.

(ii) If s^* is a w.s. for I, p is a position consistent with $s_{\hat{p}}$, and the moves in p of player II are consistent with $\vec{u}$, then $p \in \bigcap_{i<\eta} U_i$. Therefore, if s^* is a w.s. for I and x is a play consistent with $s_{\hat{p}}$, then $x \in A(\vec{u})$.

(iii) Let p be a position consistent with $s_{\hat{p}}$ and with $\vec{U}$. If the moves in p of the player for whom $s_{\hat{p}}$ is not a w.s. are consistent with $\vec{u}$, then p is consistent with $\vec{u}$.

Proof: Assume $L(\vec{U})[\#_1] \models$ "every real has a sharp" and let B, $\langle A_\alpha \mid \alpha <$

$\omega^2\rangle$, A, $\vec{U}$, and $\vec{u}$ be as in Definition 1.0. Let $R_B \in \Delta^0_1$ be such that

iv.) $B(x,n) \leftrightarrow \forall k R_B(\bar{x}(k),n)$, and

v.) if $\neg R_B(\bar{x}(k),n)$ and $\forall j < k R_B(\bar{x}(j),n)$, then k is odd.

It will be sufficient to assume $s^0 \in L(\vec{U})[\#_1]$ is a w.s. for $G^0(\vec{U};\vec{u})$ and show that s^0 can be integrated so as to obtain a w.s. $s \in L(\vec{U})[\#_1]$ for $A(\vec{U};\vec{u})$ such that (i), (ii), and (iii) holds if s^* and $s_{\hat{p}}$ are respectively replaced by s^0 and s.

Player I wins G^0 iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move. G^0 is an open game and therefore we define, for each ordinal α, P_α as the set of positions with ordinal α and let $P = \bigcup_{\alpha\in\mathrm{ON}} P_\alpha$. Also, if p is a legal position in G^0, let ℓ_p denote the set of legal positions in G^0 consistent with p. The set of legal positions for G^0 is in $L(\vec{U})[\#_1]$. Moreover, whenever p is a legal position in G^0, the following holds:

vi.) If p includes the move $\langle \hat{t}_n, t_n\rangle = \langle 1, -\rangle$, then $\ell_p \in L(T_n)$ and T_n is coded by a real in $L[\#_1]$.

Using $\langle P_\alpha | \alpha \in \mathrm{ON}\rangle$, define a wellordering $\prec$ of the legal positions for $G^0(\vec{U};\vec{u})$ such that $\prec$ is definable in $L(\vec{U})[\#_1]$ and whenever p is a legal position in $G^0(\vec{U};\vec{u})$, the following holds:

vii.) If p includes the move $\langle \hat{t}_n, t_n\rangle = \langle 1, -\rangle$, then $\prec |\ell_p$ is a wellordering of the legal positions of $G^0(\vec{U};\vec{u})$ consistent with p and is definable in $L(\vec{U}, T_n)$ from $\langle \omega_{i+1}^{L(\#_1(\vec{U},T_n))} | i \leq n\rangle$.

By Lemma 0.14, use $\prec$ to define the canonical w.s. s^0 for $G^0(\vec{U};\vec{u})$. Then s^0 is definable in $L(\vec{U})[\#_1]$. Furthermore, if p is a legal position in $G^0(\vec{U};\vec{u})$, then $s^0|\ell_p$ is a w.s. for $G^0_p(\vec{U};\vec{u})$ and is definable in any inner model of ZF in which $\prec|\ell_p$ is definable. Therefore, s^0 has the following properties:

Lemma 1.0.1. Let p be a legal position in $G^0(\vec{U};\vec{u})$. Then $s^0|\ell_p$ is a w.s. for $G^0_p(\vec{U};\vec{u})$ and if p includes the move $\langle \hat{t}_n, t_n\rangle = \langle 1, -\rangle$, then $s^0|\ell_p$ is definable in $L(\vec{U}, T_n)$ from $\langle \omega_{i+1}^{L(\#_1(\vec{U},T_n))} | i \leq n\rangle$.

If p is a legal position in $G^0(\vec{U};\vec{u})$ which includes the move $\langle \hat{t}_n, t_n\rangle = \langle 1, -\rangle$, then we use Lemma 1.0.1 and indiscernibles for $L(\vec{U}, T_n)$ to integrate $s^0|\ell_p$ with respect to the ξ_j's. Indiscernibles for $L(\vec{U}, T_n)$ exist since T_n is a real in $L(\vec{U})[\#_1]$.

Claim I: Player I has a w.s. for G_A if he has one for $G^0(\vec{U};\vec{u})$.

Let's first consider the case in which $\langle\ \rangle \in P$. Then $s^0 \in L(\vec{U})[\#_1]$ is a w.s. for I in $G^0(\vec{U};\vec{u})$. We use s^0 to define a w.s. s for I in G_A. Let $T_0 = s^0(\langle\ \rangle)$, $\langle \hat{t}_0, t_0\rangle = \langle 1, -\rangle$, and $p_1 = (T_0; \langle 1, -\rangle)$. By Lemma 1.0.1, $s^0|\ell_{p_1}$ is a w.s. for $G^0_{p_1}(\vec{U};\vec{u})$ and is definable in $L(\vec{U}, T_0)$ from $\omega_1^{L(\#_1(\vec{U},T_0))}$. Since $\vec{U}$ and T_0 can be coded by a real in $L(\vec{U})[\#_1]$ and $L(\vec{U})[\#_1] \models$ "every real has a sharp," $\#_1(\vec{U}, T_0)$ exists. Therefore, by Theorem 0.20, there exists a w.s. $s_0 \in L(\#_1(\vec{U}, T_0))$ for $A(\vec{U}, T_0; \vec{u})$. Let $s(p) = s_0(p)$ for any legal position $p = (x(0); x(1); x(2); ...; x(2i-1))$ of $A(\vec{U};\vec{u})$ such that $R_B(\bar{x}(2i), 0)$.

Otherwise, we reach a position such that

$$\neg R_B(\bar{x}(k_0), 0) \text{ and } \forall j < k_0 R_B(\bar{x}(j), 0).$$

By (iv), k_0 is odd. We have defined

$$s(x(1); x(3); ...; x(2j-1)) = x(2j) \text{ for } 2j < k_0. \tag{viii}$$

We define $s(x(1); x(3); ...; x(2j-1))$ so that it is consistent with (viii).

Disregard $\langle \hat{t}_0, t_0 \rangle = \langle 1, - \rangle$ and instead let $\langle \hat{t}_0, t_0 \rangle = \langle 0, \bar{x}(k_0) \rangle$. Let

$$p_1' = (T_0; \langle 0, \bar{x}(k_0) \rangle),\ x(0) = s^0(p_1'),$$

$$T_1 = s^0(p_1' * (x(0); x(1))), \text{ and } p_2 = p_1' * (x(0); x(1); T_1; \langle 1, - \rangle).$$

By Lemma 1.0.1, the w.s. $s^0|\ell_{p_2}$ for $G^0_{p_2}(\vec{U}; \vec{u})$ is definable in $L(\vec{U}, T_1)$ from $\langle \omega_{i+1}^{L(\#_1(\vec{U}, T_1))} | i = 0, 1 \rangle$. Since $L(\vec{U})[\#_1] \models$ "every real has a sharp," $L(\vec{U}, T_1)$ has indiscernibles. By Theorem 0.20, there exists a w.s. $s_1 \in L(\#_1(\vec{U}, T_1))$ for $A(\vec{U}, T_1; \vec{u}, \bar{x}(k_0), \bar{x}(2))$. Let $s(p) = s_1(p)$ for any legal position p of $A(\vec{U}; \vec{u})$ which extends $\bar{x}(k_0)$ and $\bar{x}(2)$.

In general, suppose we have reached a position which includes $\bar{x}(2n)$, $\bar{x}(k_0)$, $\bar{x}(k_1)$, ..., $\bar{x}(k_{n-1})$ such that $\forall i < n \neg R_B(\bar{x}(k_i), i)$. Let $\vec{t}_n = \langle \bar{x}(k_i) | i < n \rangle$ and repeat the above argument with k_0 replaced by k_{n-1}: Then we obtain s_n such that

ix.) $s_n \in L(\#_1(\vec{U}, T_n))$ is a w.s. for $A(\vec{U}, T_n; \vec{u}, \vec{t}_n, \bar{x}(2n))$,

and we let

x.) $s(p) = s_n(p)$ for any legal position p of $A(\vec{U}; \vec{u})$ which extends $\bar{x}(2n)$ and $\bar{x}(k_i)$ for all $i < n$.

Claim: The strategy s of player I is a w.s. in $G_A(\vec{U};\vec{u})$.

Let x be a play of $G_A(\vec{U};\vec{u})$ consistent with s. We show that x is a win for player I. If $B(x,n)$ for some least n, then x is consistent with s_n, and since s_n is a w.s. of I for $A(\vec{U},T_n;\vec{u},\vec{t_n},\bar{x}(2n))$, $x \in A(\vec{U};\vec{u})$ by Theorem 0.20.

So assume $\forall n \neg B(x,n)$. Then

$$y = (T_0;\langle 0,\bar{x}(k_0)\rangle;x(0);x(1);T_1;\langle 0,\bar{x}(k_1)\rangle;x(2);x(3);T_2;\langle 0,\bar{x}(k_2)\rangle;x(4);x(5);...$$
$$...;T_n;\langle 0,\bar{x}(k_n)\rangle;x(2n);x(2n+1);...)$$

is a play of $G^0(\vec{U};\vec{u})$ consistent with I's w.s. s^0 so that for some $n \in \omega$, $\bar{x}(2n)$ is inconsistent with $\vec{u}$ while $\forall j < 2n\ \bar{x}(j)$ is consistent with $\vec{u}$. Hence, $x \in A(\vec{U};\vec{u})$.

Claim II: Player II has a w.s. for G_A if he has one for $G^0(\vec{U};\vec{u})$.

Now let's consider the case $\langle\ \rangle \notin \mathrm{P}$. We integrate II's w.s. $s^0 \in L(\vec{U})[\#_1]$ for $G^0(\vec{U};\vec{u})$ to get the w.s. s for II in $G_A(\vec{U};\vec{u})$. Let

$$T_0 = \{\text{positions } p \text{ in } G_A(\vec{U};\vec{u}) \mid \forall T_0' \in L(\vec{U})[\#_1]\ \langle 0,p\rangle \neq s^0(T_0')\}$$

Then for t consistent with $\vec{U}$ and $\vec{u}$,

$$t \in T_0 \text{ iff } \forall T' \in L(\vec{U})[\#_1]\ \langle 0,t\rangle \neq s^0(T')$$
$$\text{iff } \forall T' \in L(\vec{U})[\#_1]\ (T_0';\langle 0,t\rangle) \in \textstyle\bigcup_{\alpha\in\mathrm{ON}} P_\alpha.$$

If $(T_0';\langle 0,t\rangle)$ is a legal position of $G^0(\vec{U};\vec{u})$, then by induction on α, $P_\alpha \cap \ell_{(T_0';\langle 0,t\rangle)}$ is definable in $L(\vec{U})[\#_1]$. Therefore, $T_0 \in L(\vec{U})[\#_1]$. Also, $\langle 1,-\rangle = s^0(T_0)$. Let $p_1 = (T_0;\langle 1,-\rangle)$. By Lemma 1.0.1, $s^0|\ell_{p_1}$ is a w.s. for $G^0_{p_1}(\vec{U};\vec{u})$ and is definable in $L(\vec{U},T_0)$ from $\omega_1^{L(\#_1^1(\vec{U},T_0))}$. Since $\vec{U}$ and T_0 can be coded by a real in $L(\vec{U})[\#_1]$ and $L(\vec{U})[\#_1] \models$ "every real has a sharp," $\#_1(\vec{U}, T_0)$ exists. By Theorem 0.20, let $s_0 \in L(\#_1(\vec{U}, T_0))$ be a w.s. of II for $A(\vec{U}, T_0; \vec{u})$. Let $s(p) = s_0(p)$ for any legal position $p = (x(0); x(1); x(2); ...; x(2i))$ of $A(\vec{U}; \vec{u})$ such that $\bar{x}(2i+1) \in T_0$. If we reach a position $\bar{x}(2k_0+1)$ which is not a legal position of $A(\vec{U}; \vec{u})$, then for any position p which extends $\bar{x}(2k_0+1)$, define $s(p)$ to be your favorite natural number.

Otherwise, suppose we reach a legal position $\bar{x}(2k_0+1) \notin T_0$ of $A(\vec{U}; \vec{u})$. We have defined

$$s(x(0); x(2); x(4); ...; x(2j)) = x(2j+1) \text{ for } j < k_0. \tag{xi}$$

Now we define $s(x(0); x(2); ...; x(2j))$ so that it is consistent with (xi). Since $\bar{x}(2k_0+1) \notin T_0$, there exists $T'_0 \in L(\vec{U})[\#_1]$ such that $s^0(T'_0) = \langle 0, \bar{x}(2k_0+1)\rangle$. Disregard $\langle \hat{t}_0, t_0\rangle = \langle 1, -\rangle$ and instead let $\langle \hat{t}_0, t_0\rangle = s^0(T'_0) = \langle 0, \bar{x}(2k_0+1)\rangle$. Let $p'_1 = (T'_0; \langle 0, \bar{x}(2k_0+1)\rangle; x(0))$ and define $x(1) = s(x(0))$ to be $s^0(p'_1)$. Since $x(1) = s^0(p'_1)$ must be consistent with t_0, the definition of $s(x(0))$ is consistent with (xi). Let

$$T_1 = \{\text{positions } t_1 \text{ in } G_A(\vec{U}; \vec{u}, t_0, \bar{x}(2)) | \forall T'_1 \in L[\#_1]\ \langle 0, t_1\rangle \neq s^0(T'_0; x(0); T'_1)\}.$$

Then $T_1 \in L(\vec{U})[\#_1]$, and $\langle 1, -\rangle = s^0(p'_1 * (x(1); T_1))$. Let $p_2 = p'_1 * (x(1); T_1; \langle 1, -\rangle)$. By Lemma 1.0.1, $s^0|\ell_{p_2}$ is a w.s. for $G^0_{p_2}(\vec{U}; \vec{u})$ and is definable in $L(\vec{U}, T_1)$ from $\omega_1^{L(\#_1(\vec{U}, T_1))}$ and $\omega_2^{L(\#_1(\vec{U}, T_1))}$. Since $\vec{U}$ and T_1 can be coded by a real in $L(\vec{U})[\#_1]$ and $L(\vec{U})[\#_1] \models$ "every real has a sharp," $L(\vec{U}, T_1)$ has indiscernibles. By Theorem 0.20, there exists a w.s.

$s_1 \in L(\#_1(\vec{U}, T_1))$ for $A(\vec{U}, T_1; \vec{u}, \bar{x}(2k_0{+}1), \bar{x}(2))$. Let $s(p) = s_1(p)$ for any legal position $p = (x(0); x(1); x(2); ...; x(2i))$ of $A(\vec{U}; \vec{u})$ such that $\bar{x}(2i{+}1) \in T_1$.

In general, suppose for every $i < n$, $t_i = \bar{x}(2k_i + 1)$ is consistent with s, $\bar{x}(2k_i + 1) \in (\bigcap_{j<\eta} U_j) \setminus T_i$, $\forall j < k_i\ \bar{x}(2j + 1) \in T_i \cap (\bigcap_{j<\eta} U_j)$, and the position

$p'_n = (T'_0; \langle 0, t_0\rangle; x(0); x(1); T'_1; \langle 0, t_1\rangle; x(2); x(3); ...; T'_{n-1}; \langle 0, t_{n-1}\rangle; x(2n-2))$

is consistent with s^0. Then repeat the above argument with k_0 replaced by k_{n-1}: Let $\vec{t_n} = \langle t_i | i < n\rangle$, $x(2n-1) = s^0(p'_n)$, and

$$T_n = \{\text{positions } t_n \text{ in } G_A(\vec{U}; \vec{u}; \vec{t_n}, \bar{x}(2n)) |$$
$$\forall T'_n \in L(\vec{U})[\#_1]\ \langle 0, t_n\rangle \neq s^0(p'_n * (x(2n-1); T'_n))\}.$$

Then $p_{n+1} = p'_n * (x(2n-1); T_n; \langle 1, -\rangle)$ is consistent with s^0. By Lemma 1.0.1, $s^0|\ell_{p_{n+1}}$ is a w.s. for $G^0_{p_{n+1}}(\vec{U}; \vec{u})$ and is definable in $L(\vec{U}, T_n)$ from $\langle \omega_{i+1}^{L(\#_1(\vec{U}, T_n))} | i \leq n\rangle$. Since $\vec{U}$ and T_n can be coded by a real in $L(\vec{U})[\#_1]$ and $L(\vec{U})[\#_1] \models$ "every real has a sharp," $L(\vec{U}, T_n)$ has indiscernibles. By Theorem 0.20, there exists a w.s. $s_n \in L(\#_1(\vec{U}, T_n))$ for $A(\vec{U}, T_n; \vec{u}, \vec{t_n}, \bar{x}(2n))$. Let $s(p) = s_n(p)$ for any legal position $p = (x(0); x(1); x(2); ...; x(2i))$ of $A(\vec{U}; \vec{u})$ such that $\bar{x}(2i + 1) \in T_n$.

If a position $t_n = \bar{x}(2k_n + 1)$ of least length such that $t_n \notin T_n$ is reached, then by the definition of T_n, there exists $T'_n \in L(\vec{U})[\#_1]$ such that the position $p'_{n+1} = p'_n * (x(2n-1); T'_n; \langle 0, t_n\rangle; x(2n))$ is consistent with s^0.

Claim: The strategy s of player II is a w.s. for $G_A(\vec{U}; \vec{u})$.

Let x be a play of $G_A(\vec{U};\vec{u})$ consistent with s. We show that s is a win for II.

Suppose there is a least n such that $\forall i\ \bar{x}(i) \in T_n$. Let

$$\vec{t}_n = \langle \bar{x}(2k_i+1) | i < n\rangle.$$

By the definition of s, x is consistent with the w.s. $s_n \in L(\#_1(\vec{U}, T_n))$ for $A(\vec{U}, T_n; \vec{u}, \vec{t}_n, \bar{x}(2n))$. Therefore, since s_n is a w.s. for II, $x \notin A(\vec{U};\vec{u})$ and x is a win for II in $A(\vec{U};\vec{u})$.

On the other hand, suppose for every n, there is a $t_n = \bar{x}(k_n)$ of least length such that $\bar{x}(k_n) \notin T_n$. Let $p'_0 = (\ )$. By the definition of s, obtain for each n, the position $p'_{n+1} = p'_n * (x(2n-1); T'_n; \langle 0, t_n\rangle; x(2n))$ consistent with s^0. Since p'_{n+1} is consistent with the w.s. s^0 for $G^0(\vec{U};\vec{u})$, $\neg R_B(t_n, n)$ and

xii.) if $\bar{x}(2n+1)$ is inconsistent with $\vec{u}$, then there is a $j \leq n$ such that $\bar{x}(2j+1)$ is inconsistent with $\vec{u}$ but for all $i < 2j+1$, $\bar{x}(i)$ is consistent with $\vec{u}$, which implies x is a win for II.

Since $\vec{U}$ is a sequence of I-imposed subgames of G_A,

xiii.) if x is inconsistent with $\vec{U}$, then there is a j such that $\bar{x}(2j+1)$ is inconsistent with $\vec{U}$, while for all $i < 2j+1$, $\bar{x}(i)$ is consistent with $\vec{U}$, which implies x is a win for II.

Since $\forall n \neg B(x, n)$, by (xii) and (xiii) $x \notin A(\vec{U};\vec{u})$. Consequently, s is a w.s. in $G_A(\vec{U};\vec{u})$ of the player for whom s^0 is w.s. ■

In Theorem 1.7 [Du2] we show that every $(\Pi^0_1)^*_+$ game has a w.s. in

$L(\#_2^1(0))$ if $\#_2^1(0)$ exists (i.e. if $L[\#_1]$ has indiscernibles). There, we integrate a w.s. for the G^1 auxiliary game determined by $B \in \Pi_1^0$, $\langle A_\alpha | \alpha < \omega^2 \rangle$, and $m \in \omega$. (Recall that G^1 is the game G_1^1 of §0.4.3.) Now we generalize Theorem 1.7 [Du2]. Let A be the $(\Pi_1^0)_+^*$ set witnessed by $B \in \Pi_1^0$, $\langle A_\alpha | \alpha < \omega^2 \rangle$, and $m \in \omega$. Suppose $\vec{U}$ and $\vec{u}$ respectively are sequences of I-imposed subgames of G_A and of legal positions of G_A with odd length. First we define an auxiliary game for $A(\vec{U};\vec{u})$; namely, the $G^1(\vec{U};\vec{u})$ auxiliary game determined by B, $\langle A_\alpha | \alpha < \omega^2 \rangle$, and $m \in \omega$. Then we prove that every such $A(\vec{U};\vec{u})$ has a w.s. strategy in $L(\#_2^1(\vec{U}))$ if $\#_2^1(\vec{U})$ exists.

Definition 1.1. Let $B \in \Pi_1^0$, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, and $m \in \omega$ strongly witness $A \in (\Pi_1^0)_+^*$. Let $\vec{U} = \langle U_i | i < \eta \rangle$ and $\vec{u} = \langle u_i | i < \theta \rangle$ respectively be a finite sequence of I-imposed subgames of G_A and a sequence of legal positions of G_A. Then *the* $G^1(\vec{U};\vec{u})$ *auxiliary game determined by* B, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, *and* $m \in \omega$ is the game which has exactly the same moves as G^1 and these moves are subject to the following conditions:

i.) Each $\bar{x}(i) \in \bigcap_{j<\eta} U_j$ and each $\bar{x}(i)$ must be consistent with every u_j.

ii.) $T_i \in L(\vec{U})[\#_1]$.

iii.) The λ_i's are properly ordered with repect to $\langle A_\alpha | \alpha < \omega \cdot (m+1) \rangle$ using $\langle \omega_{i+1}^{L(\#_2^1(\vec{U}))} | i \leq m \rangle$

iv.) If $\hat{t}_n = 1$, the ξ_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (n+1) \rangle$ using $\langle \omega_{i+1}^{L(\#_1^1(\vec{U},T_n))} | i \leq n \rangle$.

v.) The sequence $\langle (T_n; \langle \hat{t}_n, t_n\rangle) | \forall j < n\ \hat{t}_j = 0\rangle$ of Borel auxiliary moves and the Π^0_1 set B are related via R_B.

Player I wins $G^0(\vec{U}; \vec{u})$ iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move.

These conditions are analogous to the conditions for the moves of G^1. They are derived by changing the conditions for the moves of G^1 so that they are consistent with $\vec{U}$ and $\vec{u}$. We refer to $G^1(\vec{U}; \vec{u})$ instead of the $G^1(\vec{U}; \vec{u})$ auxiliary game determined by B, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, and $m \in \omega$ whenever B, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, and $m \in \omega$ are clear from the context. Analogous to Theorem 1.0, we have the following:

Theorem 1.1. Let B, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, m, A, $\vec{U}$, and $\vec{u}$ be as in Definition 1.1. Let $\hat{p}$ be a legal position of a game G^* such that the moves of G^* following $\hat{p}$ constitute a play of $G^1(\vec{U}; \vec{u})$. Suppose $\vec{U}$ has a definable wellordering in $L(\vec{U})$ and s^* is a w.s. for G^* such that $s^* | \ell_{\hat{p}} \in L(\vec{U})[\#_1]$. If $\#^1_2(\vec{U})$ exists, then $s^* | \ell_{\hat{p}}$ can be integrated so as to obtain a w.s. $s_{\hat{p}} \in L(\#^1_2(\vec{U}))$ for $A(\vec{U}; \vec{u})$ such that the following hold:

i.) $s_{\hat{p}}$ is a w.s. of the player for whom s^* is a w.s.,

ii.) If s^* is a w.s. for I, p is a position consistent with $s_{\hat{p}}$, and the moves in p of player II are consistent with $\vec{u}$, then $p \in \bigcap_{i<\beta} U_i$. Therefore, if s^* is a w.s. for I and x is a play consistent with $s_{\hat{p}}$, then $x \in A(\vec{u})$.

iii.) Let p be a position consistent with $s_{\hat{p}}$ and with $\vec{U}$. If the moves in p of the player for whom $s_{\hat{p}}$ is not a w.s. are consistent with $\vec{u}$, then p is consistent with $\vec{u}$.

Proof: Assume $\#_2^1(\vec{U})$ exists. Then $L(\vec{U})[\#_1] \models$ "every real has a sharp." Let B, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, m, A, $\vec{U}$, and $\vec{u}$ be as in Definition 1.1. Let $D = A^*_{\omega \cdot m}$ and let $R_B \in \Delta^0_1$ be such that

iv.) $B(x, n) \leftrightarrow \forall k R_B(\bar{x}(k), n)$, and

v.) if $\neg R_B(\bar{x}(k), n)$ and $\forall j < k\ R_B(\bar{x}(j), n)$, then k is odd.

It is sufficient to assume s^1 is a w.s. for $G^1(\vec{U}; \vec{u})$ and show that s^1 can be integrated so as to obtain a w.s. $s \in L(\#_2^1(\vec{U}))$ for $A(\vec{U}; \vec{u})$ such that (i), (ii), and (iii) holds if s^* and $s_{\hat{p}}$ are replaced by s^1 and s.

Player I wins $G^1(\vec{U}; \vec{u})$ iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move. $G^1(\vec{U}; \vec{u})$ is an open game and therefore we define, for each ordinal α, P_α as the set of positions with ordinal α and let $P = \bigcup_{\alpha \in \mathrm{ON}} P_\alpha$. Also, if p is a legal position in G^1, let ℓ_p denote the set of legal positions in $G^1(\vec{U}; \vec{u})$ consistent with p. The set of legal positions for $G^1(\vec{U}; \vec{u})$ is in $L(\vec{U})[\#_1]$. Moreover, whenever p is a legal position in $G^1(\vec{U}; \vec{u})$, the following properties hold:

vi.) If p includes the move $\langle \hat{t}_n, t_n \rangle = \langle 1, - \rangle$, then $\ell_p \in L(\vec{U}, T_n)$ and T_n is coded by a real in $L(\vec{U})[\#_1]$.

Using $\langle P_\alpha | \alpha \in \mathrm{ON} \rangle$, define a wellordering $\prec$ of the legal positions for $G^1(\vec{U}; \vec{u})$ such that $\prec$ is definable in $L(\vec{U})[\#_1]$ and whenever p is a legal position in

$G^1(\vec{U};\vec{u})$, the following holds:

vii.) If p includes the move $\langle \hat{t}_n, t_n\rangle = \langle 1, -\rangle$, then $\prec |\ell_p$ is a wellordering of the legal positions of $G^1(\vec{U};\vec{u})$ consistent with p and is definable in $L(\vec{U}, T_n)$ from $\langle \omega_{i+1}^{L(\#_1(\vec{U},T_n))} | i \leq n\rangle$.

By Lemma 0.14, use $\prec$ to define the canonical w.s. s^1 for $G^1(\vec{U};\vec{u})$. Then s^1 is definable in $L(\vec{U})[\#_1]$. Furthermore, if p is a legal position in $G^1(\vec{U};\vec{u})$, then $s^1|\ell_p$ is a w.s. for $G^1_p(\vec{U};\vec{u})$ and is definable in any inner model of ZF in which $\prec |\ell_p$ is definable. Therefore, s^1 has the following properties:

Lemma 1.1.1. Let p be a legal position in $G^1(\vec{U};\vec{u})$. Then $s^1|\ell_p$ is a w.s. for $G^1_p(\vec{U};\vec{u})$, $s^1|\ell_p \in L(\vec{U})[\#_1]$, and the following holds: If p includes the move $\langle \hat{t}_n, t_n\rangle = \langle 1, -\rangle$, then $s^1|\ell_p$ is definable in $L(\vec{U}, T_n)$ from $\langle \omega_{i+1}^{L(\#_1(\vec{U},T_n))} | i \leq n\rangle$.

If p is a legal position in $G^1(\vec{U};\vec{u})$ which includes the move $\langle \hat{t}_n, t_n\rangle = \langle 1, -\rangle$, then we use Lemma 1.1.1 and indiscernibles for $L(\vec{U}, T_n)$ to integrate $s^1|\ell_p$ with respect to the ξ_j's. We use indiscernibles for $L(\vec{U})[\#_1]$ to integrate s^1 with respect to the ordinal auxiliary moves λ_i.

Lemma 1.1.2. Let

$$p = (T_0; \langle 0, t_0\rangle; x(0), \lambda_0; x(1), \lambda_1; T_1, \langle 0, t_1\rangle; x(2), \lambda_2; x(3), \lambda_3; T_2, \langle 0, t_2\rangle; ...; x(2n-1), \lambda_{2n-1})$$

and

$$p' = (T'_0; \langle 0, t'_0\rangle; x'(0), \lambda'_0; x'(1), \lambda'_1; T'_1, \langle 0, t'_1\rangle; x'(2), \lambda'_2; x'(3), \lambda'_3; T'_2, \langle 0, t'_2\rangle; ...; x'(2n-1), \lambda'_{2n-1})$$

be legal positions of G^1 consistent with s^1.

viii.) If s^1 is a w.s. for I, the ordinal auxiliary moves λ_{2i-1} and λ'_{2i-1} are elements of C_1^1 (i.e. are indiscernibles for $L[\#_1]$) for $i \leq n$, and II's Borel auxiliary and integer moves are the same for both p and p' (i.e. $x(2i-1) = x'(2i-1)$ and $t_i = t_i'$ for $i < n$), then I's Borel auxiliary and integer moves are the same for p and p' (i.e. $x(2i) = x'(2i)$ and $T_i = T_i'$).

ix.) If s^1 is a w.s. for II, the ordinal auxiliary moves λ_{2i} and λ_{2i}' are elements of C_1^1 for $i \leq n$, and Borel auxiliary and integer moves of player I are the same for p and p' (i.e. $x(2i-2) = x'(2i-2)$ and $T_i = T_i'$), then II's Borel auxiliary and integer moves are the same for both p and p' (i.e. $x(2i+1) = x'(2i+1)$ and $t_i = t_i'$ for $i < n$).

We use Lemma 1.1.2 to show the following:

Lemma 1.1.3. Suppose for every $n \in \omega$, there exists $\langle \lambda_i^n | i < 2n \rangle$ such that $\lambda_{2i+1}^n \in C_1^1$ for $i < n$ if s^1 is a w.s. for I, $\lambda_{2i}^n \in C_1^1$ for $i < n$ if s^1 is a w.s. for II, and the position

$$p_n' = (T_0; \langle 0, t_0 \rangle; x(0), \lambda_0^n; x(1), \lambda_1^n; T_1; \langle 0, t_1 \rangle; x(2), \lambda_2^n; x(3), \lambda_3^n; T_2; \langle 0, t_2 \rangle; x(3), \lambda_3^n; x(4), \lambda_4^n; \ldots$$
$$\ldots; T_n; \langle 0, t_n \rangle)$$

is consistent with s^1. Then $x \in D(\vec{U}; \vec{u})$ iff s^1 is a w.s. for I.

We indicate how to show $x \in D(\vec{U}; \vec{u})$ if s^1 is a w.s. for I. Similarly, one shows $x \notin D(\vec{U}; \vec{u})$ if s^1 is a w.s. for II. Suppose s^1 is a w.s. for I and the positions p_n' and the sequences $\langle \lambda_i^n | i < 2n \rangle$ satisfy the hypotheses of Lemma 1.1.3. Then each $\lambda_{2i+1}^n \in C_1^1$. Use Lemma 1.1.2 and indiscernibles for $L(\vec{U})[\#_1]$ to integrate s^1 with respect to the λ_{2i+1}^n's and show $x \in D(\vec{U}; \vec{u})$.

This integration of s^1 is analogous to the integration in Theorem 0.8 [Du1] of the w.s. $\bar{s}$ (i.e. s_0^0) for $\bar{G}$ (i.e. G_0^0). In Theorem 0.8 [Du1], we use the existence of indiscernibles for L to integrate $\bar{s}$, with respect to ordinal auxiliary moves ξ_i, in such a way that we use this integration of $\bar{s}$ to show that x is in the $\omega \cdot m - \Pi_1^1$ set D when $\bar{s}$ is a w.s. for I and $x \notin D$ when $\bar{s}$ is a w.s. for II.

Now we show that $G_A(\vec{U}; \vec{u})$ has a w.s. $s \in L(\#_2^1(\vec{U}))$.

Claim I: Player I has a w.s. for $G_A(\vec{U}; \vec{u})$ if he has one for $G^1(\vec{U}; \vec{u})$.

Let's first consider the case in which $\langle\ \rangle \in P$. Then $s^1 \in L(\vec{U})[\#_1]$ is a w.s. for I in $G^1(\vec{U}; \vec{u})$. We use s^1 to define a w.s. s for I in G_A. Let $T_0 = s^1(\langle\ \rangle)$, $\langle \hat{t}_0, t_0 \rangle = \langle 1, - \rangle$, and $p_1 = (T_0; \langle 1, - \rangle)$. By Lemma 1.1.1, $s^1 | \ell_{p_1}$ is a w.s. for $G_{p_1}^1(\vec{U}; \vec{u})$ and is definable in $L(\vec{U}, T_0)$ from $\omega_1^{L(\#_1^1(\vec{U}, T_0))}$. Since $\vec{U}$ and T_0 can be coded by a real in $L(\vec{U})[\#_1]$ and $L(\vec{U})[\#_1] \models$ "every real has a sharp," $\#_1(\vec{U}, T_0)$ has indiscernibles. Therefore, by Theorem 0.20, there exists a w.s. $s_0 \in L(\#_1(\vec{U}, T_0))$ for $A(\vec{U}, T_0; \vec{u})$. Let $s(p) = s_0(p)$ for any legal position $p = (x(0); x(1); x(2); ...; x(k-1))$ of $A(\vec{U}; \vec{u})$ such that $R_B(\bar{x}(k), 0)$.

Otherwise, we reach a position such that

$$\neg R_B(\bar{x}(k_0), 0) \text{ and } \forall j < k_0 R_B(\bar{x}(j), 0).$$

By (v), k_0 is odd. We have defined

$$s(x(1); x(3); ...; x(2j-1)) = x(2j) \text{ for } 2j < k_0. \tag{x}$$

We define $s(x(1); x(3); ...; x(2j-1))$ so that it is consistent with (x).

Disregard $\langle \hat{t}_0, t_0\rangle = \langle 1, -\rangle$ and instead let $\langle \hat{t}_0, t_0\rangle = \langle 0, \bar{x}(k_0)\rangle$. Then $p'_1 = (T_0; \langle 0, \bar{x}(k_0)\rangle)$ is a legal position of $G^1(\vec{U}; \vec{u})$. Let $(x(0), \lambda_0) = s^1(p'_1)$. By Lemma 1.1.2, for every $x(1)$, there exists a unique T_1 such that the following holds: for any $\lambda_1 \in C^1_1$ in which $\langle \lambda_i | i = 0, 1\rangle$ is properly ordered with respect to $\bar{x}(2)$ and $\langle A_\alpha | \alpha < \omega \cdot (m+1)\rangle$ using $\langle \omega_{i+1}^{L(\#_2^1(\vec{U}))} | i \leq m\rangle$,

$$p'_1 * (x(0), \lambda_0; x(1), \lambda_1; T_1)$$

is consistent with s^1. Let $\langle \hat{t}_1, t_1\rangle = \langle 1, -\rangle$ and choose $\lambda_1 \in C^1_1$ so that the position

$$p_2 = p'_1 * (x(0), \lambda_0; x(1), \lambda_1; T_1; \langle 1, -\rangle),$$

is consistent with s^1. Since $\vec{U}$ and T_1 are coded by a real in $L(\vec{U})[\#_1]$ and $L(\vec{U})[\#_1] \models$ "every real has a sharp," $\#_2^1(\vec{U}, T_1)$ exists. Therefore, by Theorem 0.20, there exists a w.s. $s_1 \in L(\#_1^1(\vec{U}, T_1))$ for $A(\vec{U}, T_1; \vec{u}, t_0, \bar{x}(2))$. Let $s(p) = s_1(p)$ for any legal position $p = (x(0); x(1); x(2); ...; x(k-1))$ of $A(\vec{U}; \vec{u})$ such that $R_B(\bar{x}(k), 0)$.

In general, suppose we have reached positions $\bar{x}(k_0)$, $\bar{x}(k_1)$, ..., $\bar{x}(k_{n-1})$ such that $\forall i < n \neg R_B(\bar{x}(k_i), i)$. Then let $\vec{t_n} = \langle \bar{x}(k_i) | i < n\rangle$ and repeat the above argument with k_0 replaced by k_{n-1}:

By (v), t_i has odd length for $i < n$. By Lemma 1.1.2, for every

$$x(1), x(3), x(5), ..., x(2n-1),$$

there exists a unique $x(0), x(2), x(4), ..., x(2n-2)$ and $T_0, T_1, T_2, ..., T_n$ such that the following holds:

xi.) If $\langle \lambda'_i | i < 2n \rangle$ is properly ordered with respect to $\bar{x}(2n)$ and $\langle A_\alpha | \alpha < \omega \cdot (m+1) \rangle$ using $\langle \omega_{i+1}^{L(\#_2^1(\vec{U}))} | i \leq m \rangle$ and if $\lambda_{2i+1} = \lambda'_{2i+1} \in C_1^1$ for $i < n$, then there exists $\lambda_0, \lambda_2, \lambda_4, ..., \lambda_{2n-2}$ such that the position

$$p_n = (T_0; \langle 0, t_0 \rangle; x(0), \lambda_0; x(1), \lambda_1; T_1; \langle 0, t_1 \rangle; x(2), \lambda_2; x(3), \lambda_3; T_2; \langle 0, t_2 \rangle; x(3), \lambda_3; x(4), \lambda_4; ... \\ ...; T_n) * (\langle 1, - \rangle)$$

is consistent with s^1.

Let $\langle \hat{t}_n, t_n \rangle = \langle 1, - \rangle$ and $\vec{t}_n = \langle t_i | i < n \rangle$. Since $\vec{U}$ and T_n are coded by a real in $L(\vec{U})[\#_1]$, $\#_1^1(\vec{U}, T_n)$ exists. Therefore, by Theorem 0.20, there exists a w.s. $s_n \in L(\#_1^1(\vec{U}; T_n))$ for $A(\vec{U}, T_n; \vec{u}, \vec{t}_n, \bar{x}(2n))$. Let $s(p) = s_n(p)$ for any legal position p of $A(\vec{U}; \vec{u})$ such that $R_B(p, n)$. If we reach a position such that $\neg R_B(\bar{x}(k_n), n)$ and $\forall i < k_n R_B(\bar{x}(i), n)$, then let $\langle \hat{t}_n, t_n \rangle = \langle 0, \bar{x}(k_n) \rangle$ and choose λ_i for $i < 2n$ such that $\lambda_{2i+1} \in C_1^1$ for $i < n$ and the position

$$p'_n = (T_0; \langle 0, t_0 \rangle; x(0), \lambda_0; x(1), \lambda_1; T_1; \langle 0, t_1 \rangle; x(2), \lambda_2; x(3), \lambda_3; T_2; \langle 0, t_2 \rangle; x(3), \lambda_3; x(4), \lambda_4; ... \\ ...; T_n; \langle 0, t_n \rangle)$$

is consistent with s^1. (Note that any ordinal auxiliary move λ_i may be different for any two of the positions p_n, p'_n, p_{n+1}, and p'_{n+1}.)

By the definition of s, we have the following:

xii.) If x is consistent with s, $\forall i < n \neg B(x, i)$, and $R_B(\bar{x}(k), n)$, then $\bar{x}(k)$ is consistent with the w.s. $s_n \in L(\#_1^1(\vec{U}, T_n))$ for $A(\vec{U}, T_n; \vec{u}, \vec{t}_n, \bar{x}(2n))$, where $\vec{t}_n = \langle t_i | i < n \rangle$, $t_i = \bar{x}(k_i)$, k_i is least such that $\neg R_B(\bar{x}(k_i), i)$ (for $i < n$).

Claim: The strategy s of player I is a w.s. in $G_A(\vec{U}; \vec{u})$.

Let x be a play of $G_A(\vec{U};\vec{u})$ consistent with s. We show that x is a win for player I. If there exists a least n such that $\neg B(x,n)$, then by (xii) $x \in A(\vec{U}, T_n; \vec{u}, \vec{t}_n, \bar{x}(2n))$ so that $x \in A(\vec{U};\vec{u})$.

Otherwise, assume $\forall n \neg B(x,n)$. By the definition of s, for every n, there is a position

$$p'_n = (T_0; \langle 0, t_0\rangle; x(0), \lambda_0; x(1), \lambda_1; T_1; \langle 0, t_1\rangle; x(2), \lambda_2; x(3), \lambda_3; T_2; \langle 0, t_2\rangle; x(3), \lambda_3; x(4), \lambda_4; \ldots$$
$$\ldots; T_n; \langle 0, t_n\rangle)$$

such that p'_n is consistent with s^1 and $\lambda_{2i+1} \in C^1_1$ for $i < n$. By Lemma 1.1.3, $x \in D(\vec{U};\vec{u})$. Thus, x is a win for I.

Claim II: Player II has a w.s. for $G_A(\vec{U};\vec{u})$ if he has one for $G^1(\vec{U};\vec{u})$.

Now let's consider the case $\langle\ \rangle \notin$ P. We integrate II's w.s. $s^1 \in L(\vec{U})[\#_1]$ for $G^1(\vec{U};\vec{u})$ to get the w.s. s for II in $G_A(\vec{U};\vec{u})$. Let

$$T_0 = \{\text{positions } p \text{ in } G_A(\vec{U};\vec{u}) | \forall T'_0 \in L(\vec{U})[\#_1]\ \langle 0, p\rangle \neq s^1(T'_0)\}.$$

Then for any legal position t of $G_A(\vec{U};\vec{u})$,

$$t \in T_0 \text{ iff } \forall T'_0 \in L(\vec{U})[\#_1]\ \langle 0, t\rangle \neq s^1(T'_0)$$
$$\text{iff } \forall T'_0 \in L(\vec{U})[\#_1]\ (T'_0; \langle 0, t\rangle) \in \textstyle\bigcup_{\alpha \in \mathrm{ON}} P_\alpha.$$

If $(T'_0; \langle 0, t\rangle)$ is a legal position of $G_A(\vec{U};\vec{u})$, then by induction on α, $P_\alpha \cap \ell_{(T'_0;\langle 0,t\rangle)}$ is definable in $L(\vec{U})[\#_1]$ from $\langle \omega_{i+1}^{L(\#^1_2(\vec{U}))} | i \leq m\rangle$. Therefore, $T_0 \in L(\vec{U})[\#_1]$. Also, $\langle 1, -\rangle = s^1(\ )$. Let $p_1 = (T_0; \langle 1, -\rangle)$. By Lemma 1.1.1, $s^1 | \ell_{p_1}$ is a w.s. for $G^1_{p_1}(\vec{U};\vec{u})$ and is definable in $L(\vec{U}, T_0)$ from $\omega_1^{L(\#^1_1(\vec{U}, T_0))}$. Since $\vec{U}$ and T_0 are coded by a real in $L(\vec{U})[\#_1]$ and $L(\vec{U})[\#_1] \models$ "every real has a sharp," $\#_1(\vec{U}, T_0)$ exists. By Theorem 0.20, let $s_1 \in L(\#_1(\vec{U}))$ be a w.s. of II for $A(\vec{U}, T_0; \vec{u})$. Let $s(p) = s_1(p)$ for any legal position $p = (x(0); x(1); x(2); ...; x(2i))$ of $A(\vec{U}; \vec{u})$ such that $\bar{x}(2i+1) \in T_0$. If we reach a position $\bar{x}(2k_0+1)$ which is not a legal position of $A(\vec{U}; \vec{u})$, then for any position p which extends $\bar{x}(2k_0+1)$, define $s(p)$ to be your favorite natural number.

Otherwise, we reach a legal $\bar{x}(2k_0+1) \notin T_0$ of $A(\vec{U}; \vec{u})$. We have defined

$$s(x(0); x(2); ...; x(2j)) = x(2j+1) \text{ for } j < k_0. \tag{xiii}$$

Now we define $s(x(0); x(2); ...; x(2j))$ so that it is consistent with (xiii). Since $\bar{x}(2k_0+1) \notin T_0$, there exists $T_0' \in L(\vec{U})[\#_1]$ such that $s^1(T_0') = \langle 0, \bar{x}(2k_0+1)\rangle$. Disregard $\langle \hat{t}_0, t_0\rangle = \langle 1, -\rangle$ and instead let $\langle \hat{t}_0, t_0\rangle = s^1(T_0') = \langle 0, \bar{x}(2k_0+1)\rangle$. Choose $\lambda_0 \in C_1^1$ so that there exists a position

$$p_1' = (T_0'; \langle 0, \bar{x}(2k_0+1)\rangle; x(0), \lambda_0; x(1), \lambda_1)$$

consistent with s^1. Define $s(x(0))$ to be $x(1)$. Since $x(1)$ must be consistent with t_0, the definition of $s(x(0))$ is consistent with (xiii). Let

$$T_1 = \{\text{positions } t_1 \text{ in } G_A(\vec{U}; \vec{u}, t_0, \bar{x}(2)) |\ \forall T_1' \in L(\vec{U})[\#_1]\ \langle 0, t_1\rangle \neq s^1(p_1' * (T_1'))\}.$$

Then $T_1 \in L(\vec{U})[\#_1]$ and $\langle 1, -\rangle = s^1(p_1' * (T_1))$. Let $p_2 = p_1' * (T_1; \langle 1, -\rangle)$. By Lemma 1.1.1, $s^1|\ell_{p_2}$ is a w.s. for $G^1_{p_2}(\vec{U}; \vec{u})$ and is definable in $L(\vec{U}, T_1)$ from $\langle \omega_{i+1}^{L(\#_1(\vec{U}, T_1))} | i = 0, 1\rangle$. Since $\vec{U}$ and T_1 are coded by a real in $L(\vec{U})[\#_1]$ and $L(\vec{U})[\#_1] \models$ "every real has a sharp," $\#_1(\vec{U}, T_1)$ has indiscernibles. By Theorem 0.20, there exists a w.s. $s_1 \in L(\#_1(\vec{U}, T_1))$ for $A(\vec{U}, T_1; \vec{u}, t_0, \bar{x}(2))$. Let

$s(p) = s_1(p)$ for any legal position $p = (x(0); x(1); x(2); ...; x(2i))$ of $A(\vec{U}; \vec{u})$ such that $\bar{x}(2i+1) \in T_1$.

In general, suppose for every $i < n$,

$t_i = \bar{x}(2k_i + 1)$ is consistent with s,

$\bar{x}(2k_i + 1) \in (\bigcap_{j<\eta} U_j) \setminus T_i$,

$\forall j \leq k_i \ \bar{x}(2j) \notin T_i$,

and there exists $T'_0, T'_1, T'_2, ..., T'_{n-1}$ such that for each $i < n$, there exists a position

$$\tilde{p}_i(\langle \lambda^i_{2j} | j < i \rangle) = (T'_0; \langle 0, t_0 \rangle; x(0), \lambda^i_0; x(1), \lambda^i_1; T'_1; \langle 0, t_1 \rangle; x(2), \lambda^i_2; x(3), \lambda^i_3; ...$$
$$...; T'_i; \langle 0, t_i \rangle)$$

consistent with s^1 and such that $\forall j < i \ \lambda^i_{2j} \in C^1_1$. We define $x(2n-1) = s(x(0); x(2); x(4); ...; x(2n-2))$ by repeating the above argument with k_0 replaced by k_{n-1}:

Choose a sequence $\langle \lambda^n_{2j} | j < n \rangle$ of elements from C^1_1 for which there exists $\langle \lambda_j | j < n \rangle$ such that each $\lambda_{2j} = \lambda^n_{2j}$ and which is properly ordered with respect to $\bar{x}(2n-1)$ and $\langle A_\alpha | \alpha < \omega \cdot (m+1) \rangle$ using $\langle \omega_{i+1}^{L(\#^1_2(\vec{U}))} | i \leq m \rangle$. Since $\tilde{p}_{n-1}(\langle \lambda^{n-1}_{2j} | j < n-1 \rangle)$ is consistent with s^1, by Lemma 1.1.2 there exists $\langle \lambda^n_{2j+1} | j < n-1 \rangle$ such that $\tilde{p}_{n-1}(\langle \lambda^n_{2j} | j < n-1 \rangle)$ is the position consistent with s^1, where $\tilde{p}_{n-1}(\langle \lambda^n_{2j} | j < n-1 \rangle)$ is the position obtained from $\tilde{p}(\langle \lambda^{n-1}_{2j} | j < n-1 \rangle)$ by replacing each λ^{n-1}_j with λ^n_j. Furthermore, there exist $x(2n-1)$ and λ^n_{2n-1} such that the position $p'_n = \tilde{p}_n(\langle \lambda^n_{2j} | j <$

$n-1\rangle) * (x(2n-2), \lambda^n_{2n-2}; x(2n-1), \lambda^n_{2n-1})$ is consistent with s^1. (If $n = 0$, let $p'_n = \langle\ \rangle$.) Let $\vec{t_n} = \langle t_i | i < n\rangle$ and

$T_n = \{$positions t_n in $G_A(\vec{U}; \vec{u}, \vec{t_n}, \bar{x}(2n)) | \forall T'_n \in L(\vec{U})[\#_1]\ \langle 0, t_n\rangle \neq s^1(p'_n * (T'_n))\}$.

The position $p_{n+1} = p'_n * (T_n; \langle 1, -\rangle)$ is consistent with s^1. By Lemma 1.1.1, $s^1|\ell_{p_{n+1}}$ is a w.s. for $G^1_{p_{n+1}}(\vec{U}; \vec{u})$ and is definable in $L(\vec{U}, T_n)$ from $\langle \omega_{i+1}^{L(\#_1(\vec{U}, T_n))} | i \leq n\rangle$. Since $\vec{U}$ and T_n are be coded by a real in $L(\vec{U})[\#_1]$ and $L(\vec{U})[\#_1] \models$ "every real has a sharp," $\#_1(\vec{U}, T_n)$ exists. By Theorem 0.20, there exists a w.s. $s_n \in L(\#_1(\vec{U}, T_n))$ for $A(\vec{U}, T_n; \vec{u}, \vec{t_n}, \bar{x}(2n))$. Let $s(p) = s_n(p)$ for any legal position $p = (x(0); x(1); x(2); ...; x(2i))$ of $A(\vec{U}; \vec{u})$ such that $\bar{x}(2i+1) \in T_n$.

If we reach a position such that

$$\bar{x}(2k_n + 1) \notin (\textstyle\bigcap_{j<\eta} U_j) \setminus T_n \text{ and } \forall i' \leq 2k_n\ \bar{x}(i') \in T_n \cap (\textstyle\bigcap_{j<\eta} U_j),$$

then by the definition of T_n, there exists $T'_n \in L(\vec{U})[\#_1]$ such that the position

$$\tilde{p}_n(\langle \lambda^n_{2j} | j < n\rangle) = p'_n * (T'_n; \langle 0, \bar{x}(2k_n + 1)\rangle)$$

is consistent with s^1 and we repeat the above argument. If we reach a position $\bar{x}(k)$ such that $\bar{x}(k)$ is not a legal position of $A(\vec{U}; \vec{u})$, then for any position p which extends $\bar{x}(k)$, define $s(p)$ to be your favorite natural number.

Claim: The strategy s of player II is w.s. for $G_A(\vec{U}; \vec{u})$.

Let x be a play of $G_A(\vec{U}; \vec{u})$ consistent with s. We show that s is a win for II.

Suppose there is a least n such that $\forall i\ \bar{x}(i) \in T_n$. By the definition of s, x is consistent with the w.s. s_n for $A(\vec{U}, T_n; \vec{u}, \vec{t_n}, \bar{x}(2n))$. Since s_n is a w.s. for II, $x \notin A(\vec{U}; \vec{u})$ and x is a win for II in $A(\vec{U}; \vec{u})$.

On the other hand, suppose for every n, there is a $t_n = \bar{x}(k_n)$ of least length such that $\bar{x}(k_n) \notin T_n$. By the definition of s, for each n, obtain the position $\tilde{p}_n(\langle \lambda^n_{2j} | j < n \rangle)$ consistent with s^1 such that $\forall i < n\ \lambda^n_{2i} \in C^1_1$. By Lemma 1.1.3, $x \notin D(\vec{U}; \vec{u})$. Consequently, s is a w.s. in $G_A(\vec{U}; \vec{u})$ of the player for whom s^1 is w.s. ∎