Now we prove the natural generalization of Theorems 1.0, 1.2, and 1.4: $(\Pi^0_1, \beta * \Sigma^0_1)^*$ games are determined if $L(0^{\beta\#\frac{1}{2}})[\#_1] \models$ "every real has a sharp." The proof of this theorem is analogous to the proofs of Theorems 1.0, 1.2, and 1.4.

Theorem 1.6. If $L(0^{\beta\#\frac{1}{2}})[\#_1] \models$ "every real has a sharp," then

$$\mathrm{Det}(\Pi^0_1, \beta * \Sigma^0_1)^*.$$

Proof: Assume $L(0^{\beta\#\frac{1}{2}})[\#_1] \models$"every real has a sharp." Let

$$B \in \Pi^0_1,\ C_1, C_2, C_3, ..., C_\beta \in \Sigma^0_1, \text{ and } \langle A_\alpha \mid \alpha < \omega^2 \rangle$$

strongly witness $A \in (\Pi^0_1, \beta * \Sigma^0_1)^*$. Wlog $C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots \subseteq C_\beta$. There exist R_B, R_{C_1}, R_{C_2}, R_{C_3}, ..., R_{C_β} in Δ^0_1 such that

i.) $B(x,n) \leftrightarrow \forall k R_B(\bar{x}(k), n)$;

ii.) $C_i(x,n) \leftrightarrow \exists k R_{C_i}(\bar{x}(k), n)$;

iii.) if $\neg R_B(\bar{x}(k), n)$ and $\forall j < k R_B(\bar{x}(j), n)$, then k is odd; and

iv.) if $R_{C_i}(\bar{x}(k), n)$ and $\forall j < k \neg R_{C_i}(\bar{x}(j), n)$, then k is odd.

We show that G_A has a w.s. s. Conditions (iii) and (iv) help to simplify the proof.

We describe an open game $G^{2\beta}$ which has a w.s. $s^{2\beta} \in L(0^{\beta\#\frac{1}{2}})[\#_1]$. We integrate $s^{2\beta}$ to get the w.s. $s \in L(0^{\beta\#\frac{1}{2}})[\#_1]$ for G_A. The game $G^{2\beta}$ is similar to the auxiliary game G^4. In G^4, we initially play (for $i = 1, 2$) Borel auxiliary moves Q_i and $\langle \hat{q}_i, q_i \rangle$ which are determined by the Σ^0_1 set $\{q | \exists k R_{C_i}(q,k)\}$ (under consideration in the proof of Theorem 1.4): $G^4 \begin{array}{l} \text{I} \\ \text{II} \end{array} \begin{array}{cc} Q_2 & \\ & \langle \hat{q}_2, q_2 \rangle \end{array} \begin{array}{cc} Q_1 & \\ & \langle \hat{q}_1, q_1 \rangle \end{array} \cdots.$

Similarly, the initial moves of $G^{2\beta}$ are β pairs of Borel auxiliary moves Q_i and $\langle\hat{q}_i, q_i\rangle$:

$$G^{2\beta} \begin{array}{l} \mathrm{I} \\ \mathrm{II} \end{array} \begin{array}{cccccccccc} Q_\beta & & Q_{\beta-1} & & Q_{\beta-2} & & \cdots & Q_1 & & \\ & \langle\hat{q}_\beta, q_\beta\rangle & & \langle\hat{q}_{\beta-1}, q_{\beta-1}\rangle & & \langle\hat{q}_{\beta-2}, q_{\beta-2}\rangle & \cdots & & \langle\hat{q}_1, q_1\rangle & \cdots. \end{array}$$

For $1 \leq i \leq \beta$, $\{q | \exists k R_{C_i}(q,k)\}$ determines Q_i and $\langle\hat{q}_i, q_i\rangle$. Player II must play so that whenever

$$(Q_\beta; \langle\hat{q}_\beta, q_\beta\rangle; Q_{\beta-1}; \langle\hat{q}_{\beta-1}, q_{\beta-1}\rangle; Q_{\beta-2}; \langle\hat{q}_{\beta-2}, q_{\beta-2}\rangle; ...; Q_i; \langle\hat{q}_i, q_i\rangle)$$

is a legal position in $G^{2\beta}$, we have

v.) $$1 \leq i < j \leq n \text{ and } \hat{q}_j = 1 \Rightarrow \hat{q}_i = 1.$$

We do not allow II to play $\hat{q}_i = 0$ if $i < j$ and $\hat{q}_j = 1$ since $C_i \subseteq C_j$. Let δ be least such that $\hat{q}_\delta = 0$ or $\delta = \beta + 1$ so that the play of $G^{2\beta}$ is

$$\begin{array}{l} \mathrm{I} \\ \mathrm{II} \end{array} \begin{array}{cccccccccccccc} Q_\beta & & Q_{\beta-1} & & \cdots & Q_\delta & & Q_{\delta-1} & & Q_{\delta-2} & & \cdots & Q_1 & \\ & \langle 0, q_\beta\rangle & & \langle 0, q_{\beta-1}\rangle & \cdots & & \langle 0, q_\delta\rangle & & \langle 1, -\rangle & & \langle 1, -\rangle & \cdots & & \langle 1, -\rangle \cdots. \end{array}$$

Let $\vec{Q} = \langle Q_i | \hat{q}_i = 1\rangle$, $\vec{q} = \langle q_i | \hat{q}_i = 0\rangle$, and let μ be least such that $R_{C_\delta}(q_\delta, \mu)$ whenever $\hat{q}_\beta = 0$; otherwise μ is undefined. $G^{2\beta}$ contains a sequence $\langle (T_n; \langle\hat{t}_n, t_n\rangle) | \forall j < n\ \hat{t}_j = 0\rangle$ of Borel auxiliary moves, which is related to the Π^0_1 set B via R_B. Furthermore, player I may only play $T_i \in L(\vec{Q})[\#_1]$. If II plays $\hat{q}_\beta = 0$, then the moves of $G^{2\beta}$ following $\langle\hat{q}_1, q_1\rangle$ constitute a play of $G^1(\vec{Q}; \vec{q})$. If II plays $\hat{q}_\beta = 1$, then the moves of $G^{2\beta}$ following $\langle\hat{q}_1, q_1\rangle$ constitute a play of $G^0(\vec{Q}; \vec{q})$. If $\hat{q}_\beta = 1$ and all $\hat{t}_i = 0$, then the play of $G^{2\beta}$ is

$$G^{2\beta} \begin{array}{l} \mathrm{I} \\ \mathrm{II} \end{array} \begin{array}{ccccccc} Q_\beta & & Q_{\beta-1} & & Q_{\beta-2} & & \cdots \\ & \langle 1, -\rangle & & \langle 1, -\rangle & & \langle 1, -\rangle & \end{array}$$

$$\cdots \begin{array}{cccccccccccc} Q_1 & & T_0 & & x(0) & & T_1 & & x(2) & & T_2 & & x(4) & \\ & \langle 1, -\rangle & & \langle 0, t_0\rangle & & x(1) & & \langle 0, t_1\rangle & & x(3) & & \langle 0, t_2\rangle & & x(5) \cdots. \end{array}$$

If $\hat{q}_\beta = 0$ and all $\hat{t}_i = 0$, then the play of $G^{2\beta}$ is

$$G^{2\beta}\ \begin{array}{l} \mathrm{I} \\ \mathrm{II} \end{array} \begin{array}{cccccc} Q_\beta & & Q_{\beta-1} & & Q_{\beta-2} & \\ & \langle \hat{q}_\beta, q_\beta\rangle & & \langle \hat{q}_{\beta-1}, q_{\beta-1}\rangle & & \langle \hat{q}_{\beta-2}, q_{\beta-2}\rangle \end{array} \cdots$$

$$\cdots \begin{array}{cccccccccccc} Q_1 & & T_0 & & x(0),\lambda_0 & & T_1 & & x(2),\lambda_2 & & T_2 & & x(4),\lambda_4 & \\ & \langle \hat{q}_1, q_1\rangle & & \langle 0, t_0\rangle & & x(1),\lambda_1 & & \langle 0, t_1\rangle & & x(3),\lambda_3 & & \langle 0, t_2\rangle & & x(5),\lambda_5 \end{array} \cdots.$$

If $\hat{q}_\beta = 1$ and some $\hat{t}_n = 1$, a typical play of $G^{2\beta}$ is

$$G^{2\beta}\ \begin{array}{l} \mathrm{I} \\ \mathrm{II} \end{array} \begin{array}{cccccc} Q_\beta & & Q_{\beta-1} & & Q_{\beta-2} & \\ & \langle 1, -\rangle & & \langle 1, -\rangle & & \langle 1, -\rangle \end{array} \cdots$$

$$\cdots \begin{array}{cccccccc} Q_1 & & T_0 & & x(0) & & T_1 & & x(2) & \\ & \langle 1, -\rangle & & \langle 0, t_0\rangle & & x(1) & & \langle 0, t_1\rangle & & x(3) \end{array} \cdots$$

$$\cdots \begin{array}{cccccccccc} T_{n-1} & & x(2n-2) & & T_n & & x(2n),\xi_0 & & x(2n+2),\xi_2 & \\ & \langle 0, t_{n-1}\rangle & & x(2n-1) & & \langle 1, -\rangle & & x(2n+1),\xi_1 & & x(2n+3),\xi_3 \end{array} \cdots.$$

If $\hat{q}_\beta = 0$ and some $\hat{t}_n = 1$, a typical play of $G^{2\beta}$ is

$$G^{2\beta}\ \begin{array}{l} \mathrm{I} \\ \mathrm{II} \end{array} \begin{array}{cccccc} Q_\beta & & Q_{\beta-1} & & Q_{\beta-2} & \\ & \langle 0, q_\beta\rangle & & \langle \hat{q}_{\beta-1}, q_{\beta-1}\rangle & & \langle \hat{q}_{\beta-2}, q_{\beta-2}\rangle \end{array} \cdots$$

$$\cdots \begin{array}{cccccccccc} Q_1 & & T_0 & & x(0),\lambda_0 & & T_1 & & x(2),\lambda_2 & \\ & \langle \hat{q}_1, q_1\rangle & & \langle 0, t_0\rangle & & x(1),\lambda_1 & & \langle 0, t_1\rangle & & x(3),\lambda_3 \end{array} \cdots$$

$$\cdots \begin{array}{cccccccccc} T_{n-1} & & x(2n-2),\lambda_{2n-2} & & T_n & & x(2n),\xi_0 & & x(2n+2),\xi_2 & \\ & \langle 0, t_{n-1}\rangle & & x(2n-1),\lambda_{2n-1} & & \langle 1, -\rangle & & x(2n+1),\xi_1 & & x(2n+3),\xi_3 \end{array} \cdots.$$

If II plays $\langle \hat{q}_\beta, q_\beta\rangle = \langle 0, q_\beta\rangle$, then ordinal auxiliary moves λ_i's are played and and the λ_i's are properly ordered with respect to $\bar{x}(i+1)$ and $\langle A_\alpha | \alpha < \omega \cdot (\mu + 1)\rangle$ using $\langle \omega_{i+1}^{L(\#_2^1(\vec{Q}))} | i \leq \mu\rangle$. If II plays $\hat{t}_n = 1$, ordinal auxiliary moves ξ_i's are played, independent of what II has played for the $\hat{q}_i$'s, and the ξ_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (n+1)\rangle$ using $\langle \omega_{i+1}^{L(\#_1^1(\vec{Q}, T_n))} | i \leq n\rangle$. If II plays $\langle \hat{q}_\beta, q_\beta\rangle = \langle 0, q_\beta\rangle$ and k is least such that $R_{C_\beta}(q_\beta, k)$, then the moves of $G^{2\beta}$ following $\langle 0, q_\beta\rangle$ form a play of $G^{2\beta-1}(q_\beta)$ auxiliary game determined by B, $C_1, C_2, C_3, ..., C_\beta$, $\langle A_\alpha | \alpha < \omega^2\rangle$, and k (see Definition 1.4 which follows Theorem 1.7).

In either case, player I wins $G^{2\beta}$ iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move. $G^{2\beta}$ is an open game. Therefore, define for each ordinal α, P_α as the set of positions with ordinal α and let $P = \bigcup_{\alpha \in ON} P_\alpha$. If p is a legal position in $G^{2\beta}$, let ℓ_p denote the set of legal positions in $G^{2\beta}$ consistent with p. The set of legal positions for $G^{2\beta}$ is in $L(0^{\beta\#_2^1})[\#_1]$.

Using $\langle P_\alpha | \alpha \in \mathrm{ON}\rangle$ and Theorem 0.14, define a wellordering $\prec$ of the legal positions for $G^{2\beta}$ and the canonical w.s. $s^{2\beta}$ for $G^{2\beta}$ so that Lemma 1.6.1 (below) and the following hold: $s^{2\beta}$ is definable in $L(0^{\beta\#_2^1})[\#_1]$, and if p is a legal position in $G^{2\beta}$, then $s^{2\beta}|\ell_p$ is a w.s. for $G_p^{2\beta}$ and is definable in any inner model of ZF in which $\prec |\ell_p$ is definable. Analogous to Lemma 1.4.1, $s^{2\beta}$ has the following properties:

Lemma 1.6.1. Let p be a legal position in $G^{2\beta}$. Then $s^{2\beta}|\ell_p$ is a w.s. for $G_p^{2\beta}$ and each of the following hold:

vi.) If p includes the move $\langle \hat{q}_1, q_1\rangle$ and $\hat{q}_\beta = 1$, then $s^{2\beta}|\ell_p$ is definable in $L(\vec{Q})[\#_1]$. If p includes the move $\langle \hat{q}_1, q_1\rangle$ and $\hat{q}_\beta = 0$, then $s^{2\beta}|\ell_p$ is definable in $L(\vec{Q})[\#_1]$ from $\langle \omega_{i+1}^{L(\#_2^1(\vec{U}))} | i \leq \mu\rangle$.

vii.) If p includes the move $\langle \hat{t}_n, t_n\rangle = \langle 1, -\rangle$, then $s^{2\beta}|\ell_p$ is definable in $L(\vec{Q}, T_n)$ from $\langle \omega_{i+1}^{L(\#_1(\vec{Q}, T_n))} | i \leq n\rangle$.

viii.) If p includes the move $\langle 0, q_\beta\rangle$ and k is least such that $R_{C_\beta}(q_\beta, k)$, then $s^{2\beta}|\ell_p$ is definable in $L(0^{(\beta-1)\#_2^1})[\#_1]$ from $\langle \omega_{i+1}^{L(\beta\#_2^1(0))} | i \leq k\rangle$.

Properties (vi), (vii), and (viii) make it possible to integrate $s^{2\beta}$ in such a way that we obtain a w.s. s for G_A. Whenever p is a legal position of $G^{2\beta}$ which satisfies the hypothesis of either (vi), (vii), or (viii), we use the corresponding property (vi), (vii), or (viii) to integrate $s^{2\beta}|\ell_p$.

Claim I: Player I has a w.s. for G_A if he has one for $G^{2\beta}$.

Let's first consider the case in which $\langle\ \rangle \in P$. Then $s^{2\beta} \in L(0^{\beta\#\frac{1}{2}})[\#_1]$ is a w.s. for I in $G^{2\beta}$. We use $s^{2\beta}$ to define a w.s. $s \in L(\beta\#\frac{1}{2}(0))[\#_1]$ for I in G_A. For $1 \leq i \leq \beta$, let

$$Q_i = s^{2\beta}(Q_\beta; \langle 1, -\rangle; Q_{\beta-1}; \langle 1, -\rangle; ...; Q_{i+1}; \langle 1, -\rangle), \ \ \langle \hat{q}_i, q_i\rangle = \langle 1, -\rangle,$$

$$\text{and } p_0 = (Q_\beta; \langle 1, -\rangle; Q_{\beta-1}; \langle 1, -\rangle; ...; Q_1; \langle 1, -\rangle).$$

By Lemma 1.6.1(vi), $s^{2\beta}|\ell_{p_0}$ is definable in $L(\vec{Q})[\#_1]$. By Theorem 1.0, integrate $s^{2\beta}|\ell_{p_0}$ so as to obtain a w.s. $s_0 \in L(\vec{Q})[\#_1]$ for $A(\vec{Q})$. Let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); x(2); ...; x(i-1))$ such that $\forall j \leq \beta \forall i' \leq i \forall k \neg R_{C_j}(\bar{x}(i'), k)$.

Suppose we reach a position such that $\exists j \exists i \exists k R_{C_j}(\bar{x}(i), k)$. Since $C_j \subseteq C_\beta$ for $1 \leq j \leq \beta$, there is a least i such that $\exists k R_{C_\beta}(\bar{x}(i), k)$. By (iv), i is odd. Let k be least such that $R_{C_\beta}(\bar{x}(i), k)$. Let $Q_\beta = s^{2\beta}(\ )$ (as above), $\langle \hat{q}_\beta, q_\beta\rangle = \langle 0, \bar{x}(i)\rangle$, and $p_1 = (Q_\beta; \langle 0, q_\beta\rangle)$. By Lemma 1.6.1(viii), $s^{2\beta}|\ell_{p_1}$ is definable in $L(0^{(\beta-1)\#\frac{1}{2}})[\#_1]$ from $\langle \omega_{i+1}^{L(\beta\#\frac{1}{2}(0))} | i \leq k\rangle$. Since $L(0^{\beta\#\frac{1}{2}})$ exists, by the Induction Hypothesis, integrate $s^{2\beta}|\ell_{p_1}$ so as to obtain a w.s. $s_1 \in L(0^{\beta\#\frac{1}{2}})$ for $A(q_\beta)$. Let $s(p) = s_1(p)$ for any position p which extends q_β.

Claim: The strategy s of player I is a w.s. in G_A.

Suppose x is a play of G_A consistent with s. By the definition of s, there exists a w.s. $s_0 \in L(\vec{Q})[\#_1]$ for $A(\vec{Q})$ (where $\vec{Q} = \langle Q_i \mid 1 \leq i \leq \beta \rangle$) such that $x(2i) = s_0(x(1); x(3); ...; x(2i-1))$ whenever

$$\forall j \leq \beta \forall i' \leq 2i \forall n \neg R_{C_j}(\bar{x}(i'), n).$$

If $\forall j \leq \beta \forall i \forall n \neg R_{C_j}(\bar{x}(i), n)$ holds, then $x \in A(\vec{Q})$ so that $x \in A$ by Theorem 1.0.

Otherwise, $\exists n R_{C_\beta}(\bar{x}(i), n)$ for some least i since $C_i \subseteq C_\beta$ for $1 \leq i \leq \beta$. By the definition of s, there exists a w.s. $s_1 \in L(0^{\beta \#_2^1})$ for $A(\bar{x}(i))$ such that

$$x(2k) = s_1(x(1); x(3); ...; x(2k-1)) \text{ whenever } 2k > i.$$

Therefore, $x \in A(\bar{x}(i))$ so that $x \in A$. Thus, s is a win for I.

Claim II: Player II has a w.s. for G_A if he has one for $G^{2\beta}$.

Now let's consider the case $\langle\ \rangle \notin P$. We integrate II's w.s. $s^{2\beta}$ for $G^{2\beta}$ to get the w.s. $s \in L(\beta \#_2^1(0))[\#_1]$ for II in G_A. Let

$$Q_\beta = \{ \text{ positions } q \text{ in } G_A \mid \forall Q' \in L(\beta \#_2^1(0))[\#_1]\ \langle 0, q \rangle \neq s^{2\beta}(Q') \}.$$

Also let $Q_i = Q_\beta$ for $i < \beta$. Then $\langle 1, - \rangle = s^{2\beta}(Q_\beta)$ and $Q_i \in L(0^{\beta \#_2^1})[\#_1]$ for $1 \leq i \leq \beta$. Let $p_0 = (Q_\beta; \langle 1, - \rangle; Q_{\beta-2}; \langle 1, - \rangle; Q_{\beta-2}; \langle 1, - \rangle; ...; Q_1; \langle 1, - \rangle)$. By (v), p_0 is consistent with $s^{2\beta}$.

By Lemma 1.6.1(vii), $s^{2\beta}|\ell_{p_0}$ is a w.s. for $A(\vec{Q})$ (i.e. for $A(Q_\beta)$ since $Q_i = Q_\beta$ for $i < \beta$) and is definable in $L(Q_\beta)[\#_1]$. By Theorem 1.0, integrate $s^{2\beta}|\ell_{p_0}$ so as to obtain a w.s. $s_0 \in L(Q_\beta)[\#_1]$ for the game $A(Q_\beta)$. Let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); x(2); ...; x(i-1))$ such that

$\forall i' \leq i\ \bar{x}(i') \in Q_\beta$.

Suppose we reach a position $\bar{x}(i)$ (of least length) such that $\bar{x}(i) \notin Q_\beta$. Then i is odd and there exists $Q'_\beta \in L(0^{\beta\#\frac{1}{2}})[\#_1]$ such that the position $p_1 = (Q'_\beta; \langle 0, \bar{x}(i)\rangle)$ is consistent with $s^{2\beta}$. By Induction Hypothesis, obtain a w.s. $s_1 \in L(\beta\#^1_2(0))$ for $A(\bar{x}(i))$. Let $s(p) = s_1(p)$ for any position p which extends $\bar{x}(i)$.

Claim: The strategy s of player II is a w.s. for G_A.

Suppose x is a play of G_A consistent with s. By the definition of s, there exists a w.s. $s_0 \in L(Q_\beta)[\#_1]$ for $A(Q_\beta)$ such that

$$x(2i+1) = s_0(x(0); x(2); x(4); ...; x(2i))$$

whenever $\forall i' \leq 2i+1\ \bar{x}(i') \in Q_\beta$. If $\forall i\ \bar{x}(i) \in Q_\beta$, then $x \notin A(Q_\beta)$ so that $x \notin A$.

Otherwise, there exists (odd) i such that $\bar{x}(i) \notin Q_\beta$ and $\forall j < i\ \bar{x}(j) \in Q_\beta$. By the definition of s, there exists a w.s. $s_1 \in L(\beta\#^1_2(0))$ for $A(\bar{x}(i))$ such that $x(2k+1) = s_1(x(0); x(2); x(4); ...; x(2k))$ whenever $2k+1 \geq i$. Therefore, since s_1 is a w.s. for II, $x \notin A(\bar{x}(i))$ so that $x \notin A$. Consequently, s is a w.s. in G_A of the player for whom $s^{2\beta}$ is a w.s. ∎

Now we show that the existence of indiscernibles for $L(0^{\beta\#\frac{1}{2}})[\#^1_1]$ implies the determinacy of $(\Pi^0_1, \beta * \Sigma^0_1)^*_+$. We obtain the proof of this theorem by making changes to the proof of Theorem 1.6. These changes are analogous to the changes we made to the proof of Theorem 1.4 so as to obtain Theorem

1.5.

Theorem 1.7. If $0^{(\beta+1)\#^1_2}$ exists (i.e. $L(0^{\beta\#^1_2})[\#_1]$ has indiscernibles), then $\mathrm{Det}(\Pi^0_1, \beta * \Sigma^0_1)^*_+$.

Proof: Assume $L(0^{\beta\#^1_2})[\#_1]$ has an uncountable set $C^1_{\beta+1}$ of indiscernibles. Let

$$B \in \Pi^0_1,\ C_1, C_2, C_3, ..., C_\beta \in \Sigma^0_1,\ \langle A_\alpha \mid \alpha < \omega^2\rangle, \text{ and } m \in \omega$$

strongly witness $A \in (\Pi^0_1, \beta * \Sigma^0_1)^*_+$. Wlog $C_i \subseteq C_{i+1}$ for $i < \beta$. Then there exist R_B and R_{C_i} in Δ^0_1 (for $i \leq \beta$) such that

i.) $B(x,n) \leftrightarrow \forall k R_B(\bar{x}(k), n)$;

ii.) $C_i(x,n) \leftrightarrow \exists k R_{C_i}(\bar{x}(k), n)$;

iii.) if $\neg R_B(\bar{x}(k), n)$ and $\forall j < k R_B(\bar{x}(j), n)$, then k is odd; and

iv.) if $R_{C_i}(\bar{x}(k), n)$ and $\forall j < k \neg R_{C_i}(\bar{x}(j), n)$ for some $i \leq \beta$, then k is odd.

Conditions (iii) and (iv) help to simplify the proof.

We describe an open game $G^{2\beta+1}$ which has a w.s. $s^{2\beta+1} \in L(0^{\beta\#^1_2})[\#_1]$. We complete the proof by integrating $s^{2\beta+1}$ to get the w.s. $s \in L(0^{(\beta+1)\#^1_2})$ for G_A. The moves of $G^{2\beta+1}$ are the same as the moves of $G^{2\beta}$ with one exception: In $G^{2\beta+1}$, ordinal auxiliary moves λ_{2i} and λ_{2i+1} are respectively played with integer moves $x(2i)$ and $x(2i+1)$ whenever $\hat{q}_\beta = 1$; whereas, no ordinal auxiliary moves λ_i are played in $G^{2\beta}$ whenever II plays $\hat{q}_\beta = 1$.

If all $\hat{t}_i = 0$, then the play of $G^{2\beta+1}$ is

$G^{2\beta+1}$	I	Q_β	$Q_{\beta-1}$	$Q_{\beta-2}$	$\cdots$
	II	$\langle \hat{q}_\beta, q_\beta\rangle$	$\langle \hat{q}_{\beta-1}, q_{\beta-1}\rangle$	$\langle \hat{q}_{\beta-2}, q_{\beta-2}\rangle$	

$$\ldots \begin{array}{l} Q_2 \\ \end{array} \begin{array}{c} \\ \langle\hat q_2,q_2\rangle \end{array} \begin{array}{c} Q_1 \\ \end{array} \begin{array}{c} \\ \langle\hat q_1,q_1\rangle \end{array} \begin{array}{c} T_0 \\ \end{array} \begin{array}{c} \\ \langle 0,t_0\rangle \end{array} \begin{array}{c} x(0),\lambda_0 \\ \end{array} \begin{array}{c} \\ x(1),\lambda_1 \end{array} \begin{array}{c} T_1 \\ \end{array} \begin{array}{c} \\ \langle 0,t_1\rangle \end{array} \begin{array}{c} x(2),\lambda_2 \\ \end{array} \begin{array}{c} \\ x(3),\lambda_3 \end{array} \begin{array}{c} T_2 \\ \end{array} \begin{array}{c} \\ \langle 0,t_2\rangle \end{array} \begin{array}{c} x(4),\lambda_4 \\ \end{array} \begin{array}{c} \\ x(5),\lambda_5 \end{array} \ldots .$$

If some $\hat{t}_n = 1$, the play of $G^{2\beta+1}$ is

$$G^{2\beta+1} \begin{array}{c} \mathrm{I} \\ \mathrm{II} \end{array} \begin{array}{c} Q_\beta \\ \end{array} \begin{array}{c} \\ \langle\hat q_\beta,q_\beta\rangle \end{array} \begin{array}{c} Q_{\beta-1} \\ \end{array} \begin{array}{c} \\ \langle\hat q_{\beta-1},q_{\beta-1}\rangle \end{array} \begin{array}{c} Q_{\beta-2} \\ \end{array} \begin{array}{c} \\ \langle\hat q_{\beta-2},q_{\beta-2}\rangle \end{array} \ldots$$

$$\ldots \begin{array}{c} Q_2 \\ \end{array} \begin{array}{c} \\ \langle\hat q_2,q_2\rangle \end{array} \begin{array}{c} Q_1 \\ \end{array} \begin{array}{c} \\ \langle\hat q_1,q_1\rangle \end{array} \begin{array}{c} T_0 \\ \end{array} \begin{array}{c} \\ \langle 0,t_0\rangle \end{array} \begin{array}{c} x(0),\lambda_0 \\ \end{array} \begin{array}{c} \\ x(1),\lambda_1 \end{array} \begin{array}{c} T_1 \\ \end{array} \begin{array}{c} \\ \langle 0,t_1\rangle \end{array} \begin{array}{c} x(2),\lambda_2 \\ \end{array} \begin{array}{c} \\ x(3),\lambda_3 \end{array} \ldots$$

$$\ldots \begin{array}{c} T_{n-1} \\ \end{array} \begin{array}{c} \\ \langle 0,t_{n-1}\rangle \end{array} \begin{array}{c} x(2n-2),\lambda_{2n-2} \\ \end{array} \begin{array}{c} \\ x(2n-1),\lambda_{2n-1} \end{array} \begin{array}{c} T_n \\ \end{array} \begin{array}{c} \\ \langle 1,-\rangle \end{array} \begin{array}{c} x(2n),\xi_0 \\ \end{array} \begin{array}{c} \\ x(2n+1),\xi_1 \end{array} \begin{array}{c} x(2n+2),\xi_2 \\ \end{array} \begin{array}{c} \\ x(2n+3),\xi_3 \end{array} \ldots .$$

In either case, if II plays $\hat{q}_{i+1} = 1$, then II must play $\hat{q}_i = 1$. Whenever

$$(Q_\beta; \langle\hat{q}_\beta, q_\beta\rangle; Q_{\beta-1}; \langle\hat{q}_{\beta-1}, q_{\beta-1}\rangle; Q_{\beta-2}; \langle\hat{q}_{\beta-2}, q_{\beta-2}\rangle; ...; Q_i; \langle\hat{q}_i, q_i\rangle)$$

is a legal position in $G^{2\beta}$, we have

v.) $$1 \le i < j \le \beta \text{ and } \hat{q}_j = 1 \Rightarrow \hat{q}_i = 1.$$

We do not allow II to play $\hat{q}_i = 0$ if $i < j$ and $\hat{q}_j = 1$ since $C_i \subseteq C_j$.

Let $\vec{Q} = \langle Q_i \mid \hat{q}_i = 1\rangle$ and $\vec{q} = \langle q_i \mid \hat{q}_i = 0\rangle$. The moves of $G^{2\beta+1}$ following $\langle\hat{q}_1, q_1\rangle$ constitute a play of $G^1(\vec{Q}; \vec{q})$. If $\hat{q}_\beta = 0$, then the moves of $G^{2\beta+1}$ following $\langle\hat{q}_\beta, q_\beta\rangle$ constitute a play of $G^{2\beta-1}(q_\beta)$ (see Definition 1.4 below).

Let δ be least such that $\hat{q}_\delta = 0$ or $\delta = \beta + 1$. If $\hat{q}_i = 0$ for some i, let μ be least such that $R_{C_\delta}(q_\delta, \mu)$; otherwise, let $\mu = m$. Player I must play $Q_i \in L(0^{i\#\frac{1}{2}})$ when $i \ge \delta$, $Q_i \in L(0^{(\delta-1)\#\frac{1}{2}})$ when $i < \delta$, and $T_i \in L(\vec{Q})[\#_1]$. Otherwise, the Borel auxiliary moves must satisfy the same conditions as in $G^{2\beta}$: For $i \le \beta$, $\{q | \exists k R_{C_i}(q, k)\}$ determines the Borel auxiliary moves Q_i and $\langle\hat{q}_i, q_i\rangle$, and the sequence $\langle(T_n; \langle\hat{t}_n, t_n\rangle) | \forall j < n\ \hat{t}_j = 0\rangle$ of Borel auxiliary moves and the Π^0_1 set B are related via R_B.

The λ_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (\mu+1)\rangle$ using $\langle \omega_{i+1}^{L(\#_2^1(\vec{Q}))} | i \leq \mu\rangle$. (Notice $\#_2^1(\vec{Q})$ can equal $(\beta+1)\#_2^1(0)$.) If $\hat{t}_n = 1$, the ξ_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (n+1)\rangle$ using $\langle \omega_{i+1}^{L(\#_1^1(\vec{Q},T_n))} | i \leq n\rangle$.

Player I wins $G^{2\beta+1}$ iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move. $G^{2\beta+1}$ is an open game and therefore we define, for each ordinal α, P_α as the set of positions with ordinal α and let $P = \bigcup_{\alpha \in ON} P_\alpha$. Also, if p is a legal position in $G^{2\beta+1}$, let ℓ_p denote the set of legal positions in $G^{2\beta+1}$ consistent with p. The set of legal positions for $G^{2\beta+1}$ is in $L(0^{\beta\#_2^1})[\#_1]$.

Using $\langle P_\alpha | \alpha \in \mathrm{ON}\rangle$ and Lemma 0.14, define a wellordering $\prec$ of the legal positions for $G^{2\beta+1}$ and the canonical w.s. $s^{2\beta+1}$ for $G^{2\beta+1}$ so that Lemma 1.7.1 (below) and the following hold: $s^{2\beta+1}$ is definable in $L(0^{\beta\#_2^1})[\#_1]$, and if p is a legal position in $G^{2\beta+1}$, then $s^{2\beta+1}|\ell_p$ is a w.s. for $G_p^{2\beta+1}$ and is definable in any inner model of ZF in which $\prec |\ell_p$ is definable. Moreover, $s^{2\beta+1}$ has the following properties:

Lemma 1.7.1. Let p be a legal position in $G^{2\beta+1}$. Then $s^{2\beta+1}|\ell_p$ is a w.s. for $G_p^{2\beta+1}$ and each of the following hold:

vi.) If p includes the move $\langle 0, q_\beta\rangle$ and k is least such that $R_{C_\beta}(q_\beta, k)$, then $s^{2\beta+1}|\ell_p$ is definable in $L(0^{(\beta-1)\#_2^1})[\#_1]$ from $\langle \omega_{i+1}^{L(\beta\#_2^1(0))} | i \leq k\rangle$.

vii.) If p includes the move $\langle \hat{q}_1, q_1\rangle$, then $s^{2\beta+1}|\ell_p$ is definable in $L(\vec{Q})[\#_1]$

from $\langle \omega_{i+1}^{L(\#_2^1(\vec{Q}))} | i \leq \mu \rangle$.

viii.) If p includes the move $\langle \hat{t}_n, t_n \rangle = \langle 1, - \rangle$, then $s^{2\beta+1} | \ell_p$ is definable in $L(\vec{Q}, T_n)$ from $\langle \omega_{i+1}^{L(\#_1^1(\vec{Q}, T_n))} | i \leq n \rangle$.

If p includes the move $\langle 0, q_\beta \rangle$, then we use Properties (vi) and indiscernibles for $L(0^{(\beta-1)\#_2^1})[\#_1]$ to integrate $s^{2\beta+1} | \ell_p$ with respect to the λ_i's. If p is a legal position in $G^{2\beta+1}$ which includes the move $\langle \hat{q}_1, q_1 \rangle$, we use Properties (vii) and indiscernibles for $L(\vec{Q})[\#_1]$ to integrate $s^{2\beta+1} | \ell_p$ with respect to the λ_i's. If p is a legal position in $G^{2\beta+1}$ which includes the move $\langle \hat{t}_n, t_n \rangle = \langle 1, - \rangle$, then $\vec{Q}$ and T_n are coded by a real in $L(\vec{Q})[\#_1]$ and we use Property (viii) and indiscernibles for $L(\vec{Q}, T_n)$ to integrate $s^{2\beta+1} | \ell_p$ with resepct to the ξ_i's.

Claim I: Player I has a w.s. for G_A if he has one for $G^{2\beta+1}$.

Let's first consider the case in which $\langle \, \rangle \in P$. Then $s^{2\beta+1} \in L(0^{2\beta\#_2^1})[\#_1]$ is a w.s. for I in $G^{2\beta+1}$. We use $s^{2\beta+1}$ to define a w.s. $s \in L((\beta+1)\#_2^1(0))$ for I in G_A. Let $\langle Q_i | 1 \leq i \leq \beta \rangle$ be such that

$$p_0 = (Q_\beta; \langle 1, - \rangle; Q_{\beta-1}; \langle 1, - \rangle Q_{\beta-2}; \langle 1, - \rangle; ...; Q_1; \langle 1, - \rangle)$$

is consistent with $s^{2\beta+1}$. By Lemma 1.7.1(vii), $s^{2\beta+1} | \ell_{p_0}$ is definable in $L(\vec{Q})[\#_1]$ from $\langle \omega_{i+1}^{L(\#_2^1(\vec{Q}))} | i \leq m \rangle$. Since $L(\vec{Q})[\#_1]$ has indiscernibles, by Theorem 1.1 obtain a w.s. $s_0 \in L(\#_2^1(\vec{Q}))$ for $A(\vec{Q})$ by integrating $s^{2\beta+1} | \ell_{p_0}$. Let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); x(2); ...; x(i-1))$ such that $\forall j \leq \beta \forall i' \leq i \forall k \neg R_{C_j}(\bar{x}(i'), k)$.

Suppose we reach a position such that $\exists j \exists i \exists k R_{C_j}(\bar{x}(i), k)$. Since $C_j \subseteq C_\beta$ for $1 \leq j \leq \beta$, there is a least i such that $\exists k R_{C_\beta}(\bar{x}(i), k)$. By (iv), i is odd. Let k be least such that $R_{C_\beta}(\bar{x}(i), k)$. Let $Q_\beta = s^{2\beta+1}(\ )$ (as above), $\langle \hat{q}_\beta, q_\beta \rangle = \langle 0, \bar{x}(i) \rangle$, and $p_1 = (Q_\beta; \langle 0, q_\beta \rangle)$. By Lemma 1.7.1(vi), $s^{2\beta+1} | \ell_{p_1}$ is definable in $L(0^{(\beta-1)\#_2^1})[\#_1]$ from $\langle \omega_{i+1}^{L(\beta\#_2^1(0))} | i \leq k \rangle$. Since $0^{\beta\#_2^1}$ exists, by Induction Hypothesis (given before Theorem 1.6) there exists a w.s. $s_1 \in L(0^{\beta\#_2^1})$ for $A(q_2)$. Let $s(p) = s_1(p)$ for any position p which extends q_β.

Claim: The strategy s of player I is a w.s. in G_A.

Suppose x is a play of G_A consistent with s. By the definition of s, there exists a w.s. $s_0 \in L(\#_2^1(\vec{Q}))$ for $A(\vec{Q})$ (where $\vec{Q} = \langle Q_i \mid 1 \leq i \leq \beta \rangle$) such that $x(2i) = s_0(x(1); x(3); ...; x(2i-1))$ whenever

$$\forall j \leq \beta \ \forall i' \leq 2i \ \forall n \ \neg R_{C_j}(\bar{x}(i'), n).$$

If $\forall j \leq \beta \forall i \forall n \neg R_{C_j}(\bar{x}(i), n)$ holds, then $x \in A(\vec{Q})$ so that $x \in A$ by Theorem 1.1.

Otherwise, $\exists n R_{C_\beta}(\bar{x}(i), n)$ for some least i since $C_j \subseteq C_\beta$ for $1 \leq j \leq \beta$. By the definition of s, there exists a w.s. $s_1 \in L(0^{\beta\#_2^1})$ for $A(\bar{x}(i))$ such that

$$x(2k) = s_1(x(1); x(3); ...; x(2k-1)) \text{ whenever } 2k > i.$$

Therefore, $x \in A(\bar{x}(i))$ so that $x \in A$. Thus, x is a win for I.

Claim II: Player II has a w.s. for G_A if he has one for $G^{2\beta+1}$.

Now let's consider the case $\langle\ \rangle \notin P$. We integrate II's w.s. $s^{2\beta+1}$ for $G^{2\beta+1}$ to get the w.s. $s \in L(0^{(\beta+1)\#_2^1})$ for II in G_A. Let

$$Q_\beta = \{\text{positions } q \text{ in } G_A \mid \forall Q' \in L(\beta\#_2^1(0))[\#_1]\ \langle 0, q\rangle \neq s^{2\beta+1}(Q')\}.$$

Also let $Q_i = Q_\beta$ for $i < \beta$. Then $\langle 1, -\rangle = s^{2\beta+1}(Q_\beta)$ and $Q_i \in L(0^{2\#_2^1})[\#_1]$ for $1 \leq i \leq \beta$. Let $p_0 = (Q_\beta; \langle 1, -\rangle; Q_{\beta-2}; \langle 1, -\rangle; Q_{\beta-2}; \langle 1, -\rangle; ...; Q_1; \langle 1, -\rangle)$. By (v), p_0 is consistent with $s^{2\beta+1}$.

By Lemma 1.7.1(vii), $s^{2\beta+1}|\ell_{p_0}$ is a w.s. for $A(\vec{Q})$ (i.e. for $A(Q_\beta)$ since $Q_i = Q_\beta$ for $i < \beta$) and is definable in $L(Q_\beta)[\#_1]$ from $\langle \omega_{i+1}^{L(\#_2^1(Q_\beta))} | i \leq m\rangle$. Since $Q_\beta \in L(\beta\#_2^1(0))[\#_1]$ and $0^{(\beta+1)\#_2^1}$ exists, indiscernibles for $L(Q_\beta)[\#_1]$ exist. (The previous line is the point at which this proof breaks down when one tries to prove $\mathrm{Det}(\Pi_1^0, \beta * \Sigma_1^0)_+^*$ from a weaker determinacy hypothesis than the existence of $0^{(\beta+1)\#_2^1}$.) By Theorem 1.1, integrate $s^{2\beta+1}|\ell_{p_0}$ so as to obtain a w.s. $s_0 \in L(\#_2^1(Q_\beta)) \subseteq L(0^{(\beta+1)\#_2^1})$ for the game $A(Q_\beta)$. (Notice $\#_2^1(Q_\beta))$ can equal $0^{(\beta+1)\#_2^1}$.) Let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); x(2); ...; x(i-1))$ such that $\forall i' \leq i\ \bar{x}(i') \in Q_\beta$.

Suppose we reach a position $\bar{x}(i)$ (of least length) such that $\bar{x}(i) \notin Q_\beta$. Then i is odd and there exists $Q'_\beta \in L(0^{\beta\#_2^1})[\#_1]$ such that the position $p_1 = (Q'_\beta; \langle 0, \bar{x}(i)\rangle)$ is consistent with $s^{2\beta+1}$. By Induction Hypothesis, obtain a w.s. $s_1 \in L(\beta\#_2^1(0))$ for $A(\bar{x}(i))$ by appropriately integrating $s^{2\beta+1}|\ell_{p_1}$. Let $s(p) = s_1(p)$ for any position p which extends $\bar{x}(i)$.

Claim: The strategy s of player II is a w.s. for G_A.

Suppose x is a play of G_A consistent with s. By the definition of s, there exists a w.s. $s_0 \in L(\#_2^1(Q_\beta))$ for $A(Q_\beta)$ such that

$x(2i+1) = s_0(x(0); x(2); x(4); ...; x(2i))$ whenever $\forall i' \leq 2i+1\ \bar{x}(i') \in Q_\beta$. If $\forall i\ \bar{x}(i) \in Q_\beta$, then since s_0 is a w.s. for II, $x \notin A(Q_\beta)$ so that $x \notin A$.

Otherwise, there exists (odd) i such that $\bar{x}(i) \notin Q_\beta$ and $\forall j < i\ \bar{x}(j) \in Q_\beta$. By the definition of s, there exists a w.s. $s_1 \in L(\beta\#_2^1(0))$ for $A(\bar{x}(i))$ such that $x(2k+1) = s_1(x(0); x(2); x(4); ...; x(2k))$ whenever $2k+1 \geq i$. Since s_1 is a w.s. for II, $x \notin A(\bar{x}(i))$ so that $x \notin A$. Consequently, s is a w.s. in G_A of the player for whom $s^{2\beta+1}$ is a w.s. ∎

Definition 1.4. Let $B \in \Pi_1^0$, $C_1, C_2, C_3, ..., C_\beta \in \Sigma_1^0$, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, and $m \in \omega$ strongly witness $A \in (\Pi_1^0, \beta * \Sigma_1^0)_+^*$. Then we refer to the auxiliary game $G^{2\beta+1}$ described in the Proof of Theorem 1.7 as *the* $G^{2\beta+1}$ *auxiliary game determined by* $B \in \Pi_1^0$, $C_1, C_2, C_3, ..., C_\beta \in \Sigma_1^0$, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, *and* $m \in \omega$.

Suppose $\vec{U} = \langle U_i | i < \beta \rangle$ and $\vec{u} = \langle u_i | i < \gamma \rangle$ respectively are a finite sequence of I-imposed subgames of G_A and a sequence of legal positions of G_A. Then *the* $G^{2\beta+1}(\vec{U}; \vec{u})$ *auxiliary game determined by*

$$B,\ C_1, C_2, C_3, ..., C_\beta,\ \langle A_\alpha \mid \alpha < \omega^2 \rangle,\ \textit{and}\ m \in \omega$$

is the game in which player I wins iff a position is reached at which II cannot make a (legal) move, which has exactly the same moves as $G^{2\beta+1}$, and these moves are subject to the following conditions:

i.) $\{q | \exists k R_{C_i}(q,k)\}$ determines the Borel auxiliary moves Q_i and $\langle \hat{q}_i, q_i \rangle$ for $1 \leq i \leq \beta$.

ii.) The sequence $\langle T_n; \langle \hat{t}_n, t_n \rangle) | \forall j < n \; \hat{t}_j = 0 \rangle$ of Borel auxiliary moves and the Π^0_1 set B are related via R_B.

iii.) If $1 \leq i < j \leq \beta$ and $\hat{q}_j = 1$, then $\hat{q}_i = 1$.

iv.) Each $\bar{x}(i) \in \bigcap_{j<\beta} U_j$ and each $\bar{x}(i)$ must be consistent with every u_j.

v.) Let $\vec{Q} = \langle Q_i | \hat{q}_i = 1 \rangle$, $\vec{q} = \langle q_i | \hat{q}_i = 0 \rangle$, and δ be least such that $\hat{q}_\delta = 0$. Then $Q_i \in L(i\#^1_2(\vec{U}))[\#_1]$ for $i \geq \delta$ and $Q_i \in L((\delta - 1)\#^1_2(\vec{U}, \vec{Q}))[\#_1]$ for $i < \delta$.

vi.) $T_i \in L(\vec{U}, \vec{Q})[\#_1]$.

vii.) If $\hat{q}_\beta = 0$, let μ be least such that $R_{C_\delta}(q_\delta, \mu)$; otherwise, let $\mu = m$. The λ_i's are properly ordered with repect to $\langle A_\alpha | \alpha < \omega \cdot (\mu + 1) \rangle$ using $\langle \omega_{i+1}^{L(\#^1_2(\vec{U}, \vec{Q}))} | i \leq \mu \rangle$.

viii.) If $\hat{t}_n = 1$, the ξ_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (n+1) \rangle$ using $\langle \omega_{i+1}^{L(\#^1_1(\vec{U}, \vec{Q}, T_n))} | i \leq n \rangle$.

These conditions are analogous to the conditions for the moves of $G^{2\beta+1}$. The first three are conditions which the moves of $G^{2\beta+1}$ also must satisfy. The others are derived by changing the conditions for the moves of $G^{2\beta+1}$ so that we obtain conditions which are consistent with $\vec{U}$ and $\vec{u}$. We refer to $G^{2\beta+1}(\vec{U}; \vec{u})$ instead of the $G^{2\beta+1}(\vec{U}; \vec{u})$ auxiliary game determined by B, $C_1, C_2, C_3, ..., C_\beta$, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, and $m \in \omega$ whenever B, $C_1, C_2, C_3, ..., C_\beta$, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, and $m \in \omega$ are clear from the context. Analogous to Theorem 1.7, we have the following:

Corollary 1.7.1. Let B, $C_1, C_2, C_3, ..., C_\beta$, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, m, A, $\vec{U}$, and $\vec{u}$ be as in Definition 1.4. Let p be a legal position of a game G^* such that the moves of G^* following p constitute a play of $G^{2\beta+1}(\vec{U};\vec{u})$. Suppose $\vec{U}$ has a definable wellordering in $L(\vec{U})$, $(\beta+1)\#^1_2(\vec{U})$ exists, and s^* is a w.s. for G^* such that $s^*|\ell_p \in L(\beta\#^1_2(\vec{U}))[\#_1]$. Then $s^*|\ell_p$ can be integrated so as to obtain a w.s. $s_p \in L((\beta+1)\#^1_2(\vec{U}))$ for $A(\vec{U};\vec{u})$ such that the following hold:

i.) s_p is a w.s. of the player for whom s^* is a w.s.,

ii.) If s^* is a w.s. for I, $\hat{p}$ is a position consistent with s_p, and the moves in $\hat{p}$ of player II are consistent with $\vec{u}$, then $\hat{p} \in \bigcap_{i<\beta} U_i$. Therefore, if s^* is a w.s. for I and x is a play consistent with s_p, then $x \in A(\vec{u})$.

iii.) Let $\hat{p}$ be a position consistent with s_p and with $\vec{U}$. If the moves in $\hat{p}$ of the player for whom s_p is not a w.s. are consistent with $\vec{u}$, then $\hat{p}$ is consistent with $\vec{u}$. ∎