

Kernel and Radial Basis Methods for Boundary Value Problems of Poisson's equations in R^2

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Abstract

In this paper kernel methods proposed in Schaback [16] are described for the boundary value problems of Laplace equations in R^2 . Then, with the use of radial basis technique for particular solutions of Poisson's equation, the order of convergence of mesh free methods, by combining radial bases with kernel methods, for solving the Dirichlet problem of a Poisson's equation in an arbitrary domain of R^2 is established. Numerical examples are presented to demonstrate the efficiency of the methods.

1 Introduction

As is well known, the finite element method and finite difference method have been widely used in solving the boundary value problems, typically described as

$$\mathcal{L}u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad (1)$$

$$u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \quad (2)$$

where \mathcal{L} is a linear differential operator and Ω is a bounded and simply connected domain in R^s , $s = 2, 3$. In using these traditional methods, generating the mesh points of an arbitrary region is usually a difficult task especially for a three-dimensional problem. In the last two decades the boundary element methods have become popular due to its unique feature of mesh reduction. However, meshing a two-dimensional surface is still non-trivial. Recently mesh free methods have attracted much attention in the engineering literature (cf. [1,3-6] and references therein) for their simplicity in implementation (cf. []), i.e. neither domain nor boundary meshing is required.

In the regard of mesh free methods, Trefftz methods (cf. [6]) and the methods of fundamental solutions (MFS) (cf. [4]) are probably the ones used most for solving a

boundary value problem of a homogenous equation

$$\mathcal{L}u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \quad (3)$$

$$u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \quad (4)$$

For description, consider $\mathcal{L} = \Delta$ to be the usual Laplace operator in R^2 . Denote by \mathcal{U} the set of all harmonic functions in R^2 , or all the solutions of $\Delta u(\mathbf{x}) = 0$ in R^2 . By treating R^2 as a complex plane, all real and imaginary parts of analytic functions are harmonic. In particular, the real and imaginary parts of z^n are harmonic polynomials, expressed in polar form as $r^n \cos n\theta$ and $r^n \sin n\theta$, respectively. It is well known that $\{r^n \cos n\theta, r^n \sin n\theta; n \geq 0\}$ constitutes a complete basis of \mathcal{U} . In Trefftz methods, an approximate solution of (3) and (4) is formed in polar coordinates as

$$u_m(r, \theta) = a_0 + \sum_{n=1}^m (a_n r^n \cos n\theta + b_n r^n \sin n\theta), \quad (5)$$

where a_n, b_n are constants to be determined. Clearly u_m satisfies (3). For u_m to satisfy (4), we use the collocation methods. Namely, we choose $2m + 1$ points (r_n, θ_n) , $1 \leq n \leq 2m + 1$, on the boundary $\partial\Omega$, and set

$$u(r_n, \theta_n) = g(r_n, \theta_n), \quad 1 \leq n \leq 2m + 1,$$

which leads to a system for solving a_n and b_n . The MFS is formed or developed in a similar spirit. Instead of using harmonic polynomials, the fundamental solution of Laplace equation, given by

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \ln \|\mathbf{x} - \mathbf{y}\|, \quad \mathbf{x}, \mathbf{y} \in R^2, \quad (6)$$

is used. It is well known that

$$\Delta_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) = -\delta(\mathbf{x} - \mathbf{y}),$$

where δ is the usual Dirac delta function, and $\Delta_{\mathbf{x}}$ means that the operator is applied to the \mathbf{x} variable. Choose a fictitious domain $\tilde{\Omega}$ such that the closure $\bar{\Omega} \subset \tilde{\Omega}$, and choose m source points \mathbf{y}_k , $1 \leq k \leq m$, on $\partial\tilde{\Omega}$, then form

$$u_m(\mathbf{x}) = \sum_{k=1}^m c_k G(\mathbf{x}, \mathbf{y}_k) \quad (7)$$

where c_k , $1 \leq k \leq m$, are constants. Then $u_m(\mathbf{x})$ again satisfies (3). For u_m to satisfy (4), we use collocation methods also by choosing m collocation points \mathbf{x}_j , $1 \leq j \leq m$, on $\partial\Omega$, and setting

$$u_m(\mathbf{x}_j) = g(\mathbf{x}_j) \quad 1 \leq k \leq m \quad (8)$$

which is used to determine the coefficients $\{c_k\}$. Indeed the MFS can be viewed as a special case of Trefftz method since the set $\{G(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in \partial\tilde{\Omega}\}$ is dense in the appropriate function space on $\partial\Omega$ (cf. [3] for details).

However, both Trefftz methods and MFS lack general error analysis for arbitrary domains. Almost all convergence results of Trefftz methods and MFS are established with respect to circles (cf. []). More recently, kernel methods are borrowed in Schaback [16] to construct approximate solutions of (3) and (4) with the rate of convergence derived for arbitrary domains. Furthermore, approximate solutions of Poisson's equations (1) are explicitly constructed in [7] by radial bases that approximate the exact solutions given by Newtonian potentials. In this paper, we will combine the kernel methods [16] with radial basis methods [7] to provide approximate solutions of the boundary value problems of Poisson's equations (1) and (2), and derive the order of approximation, which our arguments will be carried out along the same line as in [9].

The organization of this paper is as follows. In Section 2 we will describe harmonic kernel methods in [16]. In Section 3 radial basis solutions of Poisson's equations in [7] will be presented. The order of convergence of kernel and radial basis methods for solving the Dirichlet problem of Poisson's equations will be derived in Section 4. Several numerical examples will be given in Section 5 to demonstrate the efficiency of the methods discussed in this paper.

2 Harmonic kernel methods for boundary value problems of Laplace equations

In this section we briefly describe the kernel methods and results in Schaback [16]. Let \mathcal{X} be a closed subset of \mathbb{R}^n . A kernel K with respect to \mathcal{X} is a function from $\mathcal{X} \times \mathcal{X}$ to \mathbb{R} . It is called a positive definite kernel, or Mercer kernel, if

$$\sum_{i,j=1}^m a_i a_j K(x_i, x_j) \geq 0, \quad (9)$$

for any set of distinct points $x_1, \dots, x_m \in \mathcal{X}$ and arbitrary constants $a_1, \dots, a_m \in \mathbb{R}$. For interesting readers [2] is referred for general theory of kernel functions.

Now assume that

$$P_\Lambda(z) := \sum_{n=0}^{\infty} \lambda_n z^n \quad (10)$$

is convergent in $B(\rho) := \{z; |z| < \rho\}$ (a disc of radius $\rho > 0$), where all the coefficients $\lambda_n > 0$ for $n \geq 0$. Its real part

$$F_\Lambda(z) := \sum_{n=0}^{\infty} \lambda_n r^n \cos(n\theta) \quad (11)$$

is harmonic for (r, θ) , $r < \rho$. For two variables (r, φ) and (s, ψ) in polar coordinates, harmonic kernels are constructed in [16] as

$$\begin{aligned} K_{\Lambda,c}((r, \varphi); (s, \psi)) &:= \sum_{n=0}^{\infty} \lambda_n c^{2n} r^n s^n \cos(n(\varphi - \psi)) \\ &= F_\Lambda(c^2 r s, \varphi - \psi) \end{aligned} \quad (12)$$

The positive definiteness of $K_{\Lambda,c}$ is verified as follows.

For arbitrary points (r_j, φ_j) , $1 \leq j \leq N$,

$$\begin{aligned} & \sum_{j,k=1}^N a_j a_k \sum_{n=0}^{\infty} \lambda_n c^{2n} r_j^n r_k^n \cos(n(\varphi_j - \varphi_k)) \\ &= \sum_{n=0}^{\infty} \lambda_n c^{2n} \sum_{j,k=1}^N a_j a_k r_j^n r_k^n (\cos(n\varphi_j) \cos(n\varphi_k) + \sin(n\varphi_j) \sin(n\varphi_k)) \end{aligned} \quad (13)$$

$$= \sum_{n=0}^{\infty} \lambda_n c^{2n} \left(\left(\sum_{j=1}^N a_j r_j^n \cos(n\varphi_j) \right)^2 + \left(\sum_{j=1}^N a_j r_j^n \sin(n\varphi_j) \right)^2 \right) \geq 0 \quad (14)$$

If it is 0, then

$$\sum_{j=1}^N a_j r_j^n \cos(n\varphi_j) = 0, \text{ and } \sum_{j=1}^N a_j r_j^n \sin(n\varphi_j) = 0, \quad (15)$$

for all $n \geq 0$. Thus for $z_j = (r_j, \varphi_j)$, $1 \leq j \leq N$ and $n \geq 0$

$$\sum_{j=1}^N a_j z_j^n = 0, \quad (16)$$

which implies $a = \{a_j\}_{j=1}^N = 0$. Below we write out two specific harmonic kernels.

Choose $\lambda_n = \frac{1}{n!}$, and use the identity

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z) = \exp(r \cos \theta + ir \sin \theta), \quad (17)$$

then by comparing the real parts of both sides of the above equation, we conclude

$$\sum_{n=0}^{\infty} \frac{r^n \cos n\theta}{n!} = \exp(r \cos \theta) \cdot \cos(r \sin \theta). \quad (18)$$

Hence a harmonic ‘‘exponential’’ kernel is formed as

$$\begin{aligned} K_{\Lambda,c}((r, \varphi); (s, \psi)) &:= \sum_{n=0}^{\infty} \frac{1}{n!} c^{2n} r^n s^n \cos(n(\varphi - \psi)) \\ &= \exp(c^2 r s \cos(\varphi - \psi)) \cdot \cos(c^2 r s \sin(\varphi - \psi)) \end{aligned} \quad (19)$$

Another type of harmonic kernels is constructed from the meromorphic function $\frac{1}{1-z}$. With $\lambda_n = 1$,

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} = \frac{1}{1-r \cos \theta - ir \sin \theta}.$$

Computing the real parts of both sides, we have

$$\sum_{n=0}^{\infty} r^n \cos n\theta = \frac{1 - r \cos \theta}{1 - 2r \cos \theta + r^2} \quad (20)$$

This yields the Poisson kernel

$$\begin{aligned} K_{\Lambda,c}((r, \varphi); (s, \psi)) &:= \sum_{n=0}^{\infty} c^{2n} r^n s^n \cos(n(\varphi - \psi)) \\ &= \frac{1 - c^2 r s \cos(\varphi - \psi)}{1 - 2c^2 r s \cos(\varphi - \psi) + c^4 r^2 s^2} \end{aligned} \quad (21)$$

which requires $c^2 r s < 1$.

Now suppose that we have chosen a harmonic kernel $K(\mathbf{x}, \mathbf{y})$, and a set of points \mathbf{x}_k , $1 \leq k \leq m$, on $\partial\Omega$, where $\Omega \subset B(\rho)$. Then any function

$$u_m(\mathbf{x}) = \sum_{i=1}^m c_i K(\mathbf{x}, \mathbf{x}_i), \quad \mathbf{x} \in \Omega, \quad (22)$$

where $\{c_i\}$ are arbitrary constants, is harmonic in Ω . To determine $\{c_i\}$, we use the collocation method as described in (8). Suppose that $u(\mathbf{x})$ is the exact solution of (3)-(4). To estimate the approximation error $u(\mathbf{x}) - u_m(\mathbf{x})$, from the maximum principle for harmonic functions, we only need to estimate $u(\mathbf{x}) - u_m(\mathbf{x})$, or $g(\mathbf{x}) - u_m(\mathbf{x})$, on the boundary $\partial\Omega$. Assume that $\partial\Omega$ is parameterized by

$$\mathbf{x}(t) = \rho(t)(\cos t, \sin t), \quad 0 \leq t \leq 2\pi. \quad (23)$$

For the sake of convenience, we say $g(\mathbf{x}) \in C^{(k)}(\partial\Omega)$ if $g(t) := g(\rho(t)(\cos t, \sin t))$, as a function of t , is in $C^k[0, 2\pi]$, and $\|g^{(k)}\|_{L^\infty(\partial\Omega)}$ represents $\|g^{(k)}(t)\|_{L^\infty([0, 2\pi])}$. Let $t_k = \frac{2\pi}{m}k$, $1 \leq k \leq m$, and

$$\mathbf{x}_k = \mathbf{x}(t_k) = \rho(t_k)(\cos t_k, \sin t_k). \quad (24)$$

Then the following result is proved in [16].

Theorem 1 *Suppose that $g(\mathbf{x}) \in C^{(k)}(\partial\Omega)$, choose λ_n and c such that*

$$\sum_{n=0}^{\infty} \lambda_n c^{2n} n^{2k} < \infty. \quad (25)$$

Let u be the exact solution of (3)-(4), and u_m an approximate solution determined by (22) and (8). Then

$$\|u - u_m\|_{L^\infty(\Omega)} \leq \frac{c}{m^{k-1/2}} \|g^{(k)}\|_{L^\infty(\partial\Omega)}. \quad (26)$$

3 Particular solutions of Poisson's equation by radial bases

To solve the boundary value problem of a Poisson's equation, we first need to find particular solutions of Poisson's equation, and then reduce it to the boundary value problem of a Laplace equation, as shown in Section 4. For this purpose, we describe the methods in [7] to obtain particular solutions of Poisson's equation below by radial bases

$$\Delta u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega. \quad (27)$$

Let Z^2 be the set of all multi-integers in R^2 . For $\delta > 0$, let $\Omega_\delta = \Omega + \delta I$, where $I = [-1, 1]^2$, and

$$I_n(\Omega_\delta) = \left\{ \mathbf{j} \in Z^2; \left[\frac{\mathbf{j}}{n}, \frac{\mathbf{j} + \mathbf{1}}{n} \right]^2 \cap \Omega_\delta \neq \emptyset \right\} \quad (28)$$

with $\mathbf{1} = (1, 1) \in Z^2$. For an integer $m \geq 0$, denote by $C^m(\Omega)$ the space of all functions f whose partial derivatives $D^{\mathbf{k}}f$ for $|\mathbf{k}| \leq m$ are bounded and continuous in Ω with

$$\|f\|_{C^m(\Omega)} = \sum_{|\mathbf{k}| \leq m} \|D^{\mathbf{k}}f\|_{C(\Omega)}. \quad (29)$$

When $m = 0$, we write $C^0(\Omega) = C(\Omega)$. For a function $f \in C^m(\Omega_\delta)$, one can choose a smooth function χ such that χ is identical to 1 on the closure $\bar{\Omega}$, and vanishes outside of Ω_δ . Let $f_\chi = f \cdot \chi$, then $f_\chi \in C^m(R^2)$ and it is compactly supported in $\bar{\Omega}_\delta$. Denote by $C_0^m(\Omega_\delta)$ the subspace of functions in $C^m(R^2)$ which vanish outside of Ω_δ . We then consider the approximation of functions in $C_0^m(\Omega_\delta)$ over the domain Ω .

Choose γ such that $0 < \gamma < 1$. For every $f \in C_0^m(\Omega_\delta)$ and an integer $n \geq 1$, the following approximation operator is introduced in [14].

$$\mathcal{B}_n f(x) = \frac{1}{n^{2(1-\gamma)}} \sum_{\mathbf{j} \in I_n(\Omega_\delta)} f\left(\frac{\mathbf{j}}{n}\right) \phi(n^\gamma \mathbf{x} - \mathbf{j}n^{\gamma-1}), \quad (30)$$

which is generalized in [8] to arbitrarily scattered data with more accurate order of approximating functions and their derivatives derived.

Suppose that $\phi \in C^{m+1}(R^2)$ satisfies

$$\int_{R^2} \phi(\mathbf{x}) d\mathbf{x} = 1, \quad (31)$$

and

$$\left| D^{\mathbf{k}}\phi(\mathbf{x}) \right| \leq c(1 + \|\mathbf{x}\|)^{-\alpha}, \quad |\mathbf{k}| \leq m + 1 \quad (32)$$

for some $\alpha > 2$. Among other approximation results, it is shown in [8] that when $f \in C_0^1(\Omega_\delta)$, $\phi \in C^1(R^2)$ satisfying (31) and (32) for some $\alpha > 3$, then

$$\|\mathcal{B}_n f - f\|_{C(\Omega)} \leq c \left(\frac{1}{n^\gamma} + \frac{1}{n^{1-\gamma}} \right) \|f\|_{C^1(\Omega_\delta)}, \quad (33)$$

which clearly the optimal order of approximation is achieved at $\gamma = 1/2$.

Now choose ϕ to be a radial basis function, i.e. $\phi(\mathbf{x}) = \phi(r)$, where $r = \|\mathbf{x}\|$. Then a radially particular solution of

$$\Delta\psi(r) = \phi(r)$$

is given in [7] by

$$\psi(r) = \left(\int_0^r t\phi(t)dt \right) \ln r - \int_0^r t\phi(t) \ln t dt \quad (34)$$

Set

$$u_n(\mathbf{x}) = \frac{1}{n^2} \sum_{\mathbf{j} \in I_n(\Omega_\delta)} f\left(\frac{\mathbf{j}}{n}\right) \psi(n^\gamma(\mathbf{x} - \mathbf{j}/n)). \quad (35)$$

Then it is easy to see

$$\Delta u_n(\mathbf{x}) = \mathcal{B}_n f(\mathbf{x}),$$

or $u_n(\mathbf{x})$ serves as an approximate solution of (27). To derive the main result of this paper in section 4, we need to estimate the derivatives of u_n below.

Lemma 1 *Suppose that $\phi(r)$ satisfies (31) and (32), $\psi(r)$ is given as in (34). Let u_n be an approximate particular solution of (27) given by (35). Then for any $0 \leq j \leq k$, where $k \leq m$,*

$$\|u_n^{(j)}\|_{L^\infty(\partial\Omega)} \leq cn^{(j+2)\gamma}. \quad (36)$$

Proof. From (34), it is easy to get

$$\psi(r) = -r^2 \int_0^1 s \ln s \phi(sr) ds. \quad (37)$$

It is clear from (32) that

$$|D^{\mathbf{j}}\phi(sr)| \leq c, \quad 0 \leq |\mathbf{j}| \leq k \quad (38)$$

where $0 \leq s \leq 1$, and thus we get from (37)

$$D^{\mathbf{j}}\psi(r) = O(r^2) \quad (39)$$

Observe

$$D^{\mathbf{j}}[\psi(n^\gamma(\mathbf{x} - \mathbf{k}/n))] = n^{|\mathbf{j}|\gamma} D^{\mathbf{j}}\psi(n^\gamma(\mathbf{x} - \mathbf{k}/n)) \quad (40)$$

which shows

$$|D^{\mathbf{j}}[\psi(n^\gamma(\mathbf{x} - \mathbf{k}/n))]| \leq cn^{(|\mathbf{j}|+2)\gamma} \quad (41)$$

Hence

$$|D^{\mathbf{j}}u_n(\mathbf{x})| \leq \frac{c}{n^2} \sum_{\mathbf{j} \in I_n(\Omega_\delta)} \|f\|_{L^\infty(\Omega_\delta)} n^{(|\mathbf{j}|+2)\gamma} \leq cn^{(|\mathbf{j}|+2)\gamma} \quad (42)$$

Furthermore

$$\|u_n^{(j)}\|_{L^\infty(\partial\Omega)} \leq c \max_{\mathbf{j} \in \mathbb{Z}_+^2, |\mathbf{j}|=j} \|D^{\mathbf{j}}u_n\|_{L^\infty(\partial\Omega)} \quad (43)$$

Therefore the estimate of (36) follows.

4 Convergence of solving Poisson's equation by radial bases and kernel methods

For a boundary value problem of a Poisson's equation

$$\Delta u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad (44)$$

$$u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \quad (45)$$

the results on the existence and uniqueness of the solution are well known or established in the literature under certain suitable conditions. In this paper we propose numerical methods to solve the above boundary value problem by combining the kernel and radial basis methods discussed in Sections 2 and 3. To be precise, we first find an approximate solution u_n of $\Delta u_n(\mathbf{x}) = B_n f(\mathbf{x})$, and then solve the following problem

$$\Delta v(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \quad (46)$$

$$v(\mathbf{x}) = g(\mathbf{x}) - u_n(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \quad (47)$$

by the kernel methods, which we denote the solution by $v_{n,m,k}$. Set

$$u_{n,m,k} = u_n + v_{n,m,k}, \quad (48)$$

which is considered as an approximate solution of (44) and (45). The numerical examples of computing $u_{n,m,k}$ and comparing them with the exact solutions will be presented in Section 5. Here we first prove a theoretic result on the order of approximating exact solution u by $u_{n,m,k}$.

Theorem 2 *Let u be the exact solution of (44) and (45), where $f \in C_0^m(\Omega_\delta)$, $g \in C^k(\partial\Omega)$, and $u_{n,m,k}$ an approximate solution given by (48). Then*

$$\|u - u_{n,m,k}\|_{L^\infty(\Omega)} \leq c \left(\frac{1}{n^\gamma} + \frac{1}{n^{1-\gamma}} \right) \|f\|_{C^1(\Omega_\delta)} + \frac{c}{m^{k-1/2}} n^{(k+2)\gamma}. \quad (49)$$

Proof. Let v_n be the exact solution of

$$\Delta v(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \quad (50)$$

$$v(\mathbf{x}) = g(\mathbf{x}) - u_n(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \quad (51)$$

Set

$$\tilde{u}_n = u_n + v_n \quad (52)$$

Then

$$\|u_{n,m,k} - u\|_{L^\infty(\Omega)} \leq \|u_{n,m,k} - \tilde{u}_n\|_{L^\infty(\Omega)} + \|\tilde{u}_n - u\|_{L^\infty(\Omega)} \quad (53)$$

From Theorem 1 and Lemma 1, we have

$$\begin{aligned} \|u_{n,m,k} - \tilde{u}_n\|_{L^\infty(\Omega)} &= \|v_n - v_{n,m,k}\|_{L^\infty(\Omega)} \\ &\leq \frac{c}{m^{k-1/2}} \|(g - u_n)^{(k)}\|_{L^\infty(\partial\Omega)} \\ &\leq \frac{c}{m^{k-1/2}} n^{(k+2)\gamma}. \end{aligned} \quad (54)$$

Note that for $\mathbf{x} \in \Omega$,

$$\Delta(\tilde{u}_n - u) = \Delta u_n + \Delta v_n - \Delta u = B_n f - f \quad (55)$$

and thus from (33),

$$\|\Delta(\tilde{u}_n - u)\|_{L^\infty(\Omega)} \leq c \left(\frac{1}{n^\gamma} + \frac{1}{n^{1-\gamma}} \right) \|f\|_{C^1(\Omega_\delta)} \quad (56)$$

When $\mathbf{x} \in \partial\Omega$,

$$\tilde{u}_n(\mathbf{x}) - u(\mathbf{x}) = u_n(\mathbf{x}) + g(\mathbf{x}) - u_n(\mathbf{x}) - g(\mathbf{x}) = 0 \quad (57)$$

It follows from (56) and (57) and a-priori estimate that

$$\|\tilde{u}_n - u\|_{L^\infty(\Omega)} \leq c \left(\frac{1}{n^\gamma} + \frac{1}{n^{1-\gamma}} \right) \|f\|_{C^1(\Omega_\delta)} \quad (58)$$

Hence the conclusion of the theorem follows from (53), (54) and (58).

5 Numerical examples

Let us first present an example of solving the boundary value problem of a Laplace equation by the kernel methods in Section 2. We also compute the numerical solutions by Trefftz methods and MFS, as described in Section 1, for the purpose of comparison.

Example 1 Consider the Dirichlet problem of a Laplace equation

$$\Delta u(x, y) = 0, \quad (x, y) \in \Omega, \quad (59)$$

$$u(x, y) = e^{x^2-y^2} \cos(2xy), \quad (x, y) \in \partial\Omega, \quad (60)$$

where the exact solution is $u(x, y) = e^{x^2-y^2} \cos(2xy)$. The boundary of Ω is parameterized by $x(t) = \frac{\cos^2 t}{2+\cos t}$, $y(t) = \frac{\sin t}{2+\cos t}$, $0 \leq t < 2\pi$, (see Fig.1). Set $t_k = \frac{2\pi(k-1)}{m}$, $1 \leq$

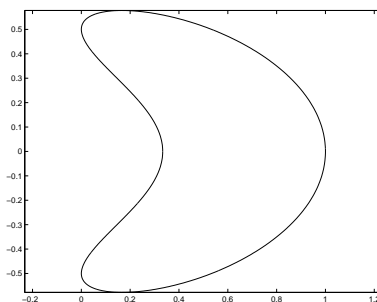


Figure 1: Graph of $\partial\Omega$.

$k \leq m$, and choose the collocation points

$$\mathbf{x}_k = \left(\frac{\cos^2 t_k}{2 + \cos t_k}, \frac{\sin t_k}{2 + \cos t_k} \right) \in \partial\Omega. \quad (61)$$

For the MFS, we choose the fictitious boundary $\partial\tilde{\Omega} = \{x^2 + y^2 = 3\}$ and the source points $(3 \cos t_k, 3 \sin t_k)$, $1 \leq k \leq m$, on $\partial\tilde{\Omega}$. Since the maximum value principle applies to harmonic functions, we only need to test the approximate errors on the boundary. With respect to $g(x, y) = e^{x^2 - y^2} \cos(2xy)$, let $u_m(\mathbf{x})$ be the numerical solution computed by Trefftz method (5), MFS (7), and kernel method (22), respectively, in which only exponential kernel (19) is used. We evaluate $e(\mathbf{x}) = |u_m(\mathbf{x}) - u(\mathbf{x})|$ or $|u_m(\mathbf{x}) - g(\mathbf{x})|$ on 1000 points of $\partial\Omega$. Table 1 lists out the numerical results.

Table 1: $\|e\|_{L^\infty(\partial\Omega)}$.

| m | $m = 16$ | $m = 20$ | $m = 24$ |
|-------------|---------------|---------------|---------------|
| kernel(exp) | $3.0498e - 6$ | $1.5715e - 7$ | $4.4091e - 8$ |
| Trefftz | $5.2723e - 6$ | $1.9778e - 7$ | $5.4402e - 9$ |
| MFS | $7.8732e - 6$ | $6.9710e - 7$ | $3.5233e - 8$ |

It shows that all three methods perform similarly for a Laplace equation, which turns out to be different for Poisson's equations, as shown in the last example.

To use radial basis methods for Poisson's equations, we need to construct a smooth compactly supported function, which we describe a truncation function in [12]. For a given interval $[a, b]$, let

$$p(x) = \sum_{k=0}^n \frac{(-1)^{n-k}}{2n - k + 1} \binom{n}{k} (b - a)^k (x - a)^{2n - k + 1}. \quad (62)$$

Define

$$T(a, b; x) = \begin{cases} 0, & x \leq a, \\ p(x)/p(b), & a < x < b, \\ 1, & b \leq x. \end{cases} \quad (63)$$

Then $T(a, b; x)$ is n -times continuously differentiable, equals 0 for $x \leq a$ and 1 when $x \geq b$, and is a polynomial of degree $2n + 1$ for $x \in [a, b]$. For a given $d > 0$, let

$$V(a, b, d; x) = T(a - d, a; x)T(-b - d, -b; -x). \quad (64)$$

Then V takes on 1 in $[a, b]$, 0 outside of $[a - d, b + d]$, and is n -times continuously differentiable.

We next consider a Poisson's equation by combining kernel and radial basis methods.

Example 2 Consider the Dirichlet problem of a Poisson's equation

$$\Delta u(x, y) = e^x + 2, \quad (x, y) \in \Omega, \quad (65)$$

$$u(x, y) = e^x + y^2, \quad (x, y) \in \partial\Omega, \quad (66)$$

where $\Omega = \{(x-1/2)^2 + (y-1/2)^2 \leq (1/2)^2\}$, and the exact solution is $u(x, y) = e^x + y^2$. Let $f(x, y) = e^x + 2$, and f_χ be given by

$$f_\chi(x, y) = f(x, y)V(0, 1, d; x)V(0, 1, d; y). \quad (67)$$

The profiles of $V(0, 1, 0.3; x)V(0, 1, 0.3; y)$ and f_χ are shown in Fig.2. $\mathcal{B}_n f_\chi$ is then

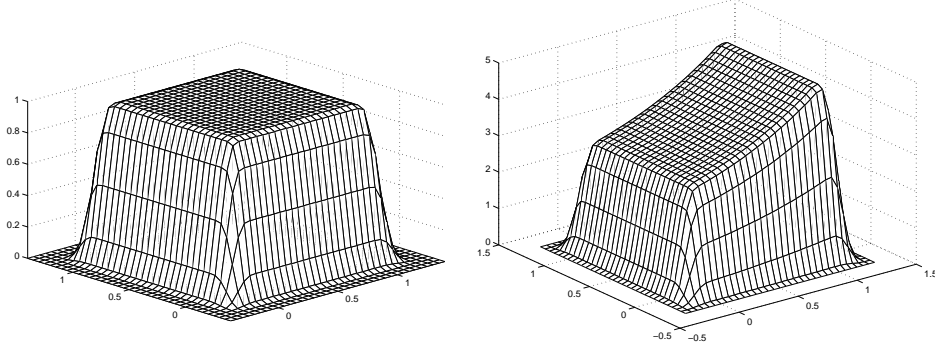


Figure 2: Profile of $V(0, 1, 0.3; x)V(0, 1, 0.3; y)$ and f_χ .

expressed as

$$\mathcal{B}_n f_\chi(x, y) = \frac{1}{n^{2(1-\gamma)}} \sum_{i=-(1+d)n}^{(1+d)n} \sum_{j=-(1+d)n}^{(1+d)n} f_\chi\left(\frac{i}{n}, \frac{j}{n}\right) \phi\left(n^\gamma \left(x - \frac{i}{n}, y - \frac{j}{n}\right)\right), \quad (68)$$

where we use

$$\phi(r) = \begin{cases} \frac{5}{\pi}(1-r^2)^4, & 0 \leq r \leq 1, \\ 0, & r \geq 1. \end{cases} \quad (69)$$

As an illustration, the approximation error $|\mathcal{B}_n f_\chi(\mathbf{x}) - f(\mathbf{x})|$ is shown in Table 2 with respect to different choices of γ , which is done by numerically evaluating the error at 40×40 uniform grid points in $[0, 1]^2$ with $n = 100$, $d = 0.3$. Correspondingly the

Table 2: $\|\mathcal{B}_n f_\chi - f\|_{L^\infty(\Omega)}$.

| $\gamma = \frac{1}{3}$ | $\gamma = \frac{1}{2}$ | $\gamma = \frac{2}{3}$ | $\gamma = \frac{3}{4}$ |
|------------------------|------------------------|------------------------|------------------------|
| 0.1977 | $9.2687e - 3$ | $4.9310e - 4$ | $3.6920e - 3$ |

particular solutions of $\Delta u = \mathcal{B}_n f_\chi$ are then given by

$$u_n(x, y) = \frac{1}{n^2} \sum_{i=-(1+d)n}^{(1+d)n} \sum_{j=-(1+d)n}^{(1+d)n} f_\chi\left(\frac{i}{n}, \frac{j}{n}\right) \psi\left(n^\gamma \left(x - \frac{i}{n}, y - \frac{j}{n}\right)\right), \quad (70)$$

where ψ is given by (34). We now use the kernel method to solve the following boundary value problem, in which both exponential kernel (19) and Poisson's kernel (21) are used,

$$\Delta v(x, y) = 0, \quad (x, y) \in \Omega, \quad (71)$$

$$v(x, y) = e^x + y^2 - u_n(x, y), \quad (x, y) \in \partial\Omega. \quad (72)$$

Denote the numerical solution of (71)-(72) by $v_{n,m}$, and set

$$u_{n,m} = u_n + v_{n,m}, \quad (73)$$

which serves as an approximate solution of Eqs (65) and (66). Let

$$e(\mathbf{x}) = |u(\mathbf{x}) - u_{n,m}(\mathbf{x})| \quad (74)$$

be the approximate error. To evaluate $\|e\|_{L^\infty(\Omega)}$ numerically, we use 40×40 equally distributed points in $[0, 1]^2$. Then we justify these points. If a point is in Ω , we evaluate e at that point, otherwise we consider $e = 0$. Table 3 shows the maximal absolute errors. The profiles of exact solution and approximate solution using $m = 24$ are shown in Fig.3. The profile of errors is shown in Fig.4.

Table 3: $\|e\|_{L^\infty(\Omega)}$ on domain of circle.

| m | $m = 12$ | $m = 16$ | $m = 20$ | $m = 24$ | $m = 30$ |
|-------------|---------------|---------------|---------------|---------------|---------------|
| kernel(exp) | $1.6506e - 4$ | $9.3619e - 6$ | $4.4380e - 6$ | $4.4311e - 6$ | $4.4345e - 6$ |
| kernel(poi) | 0.1817 | $4.5749e - 2$ | $1.3036e - 2$ | $3.9563e - 3$ | $5.5887e - 4$ |

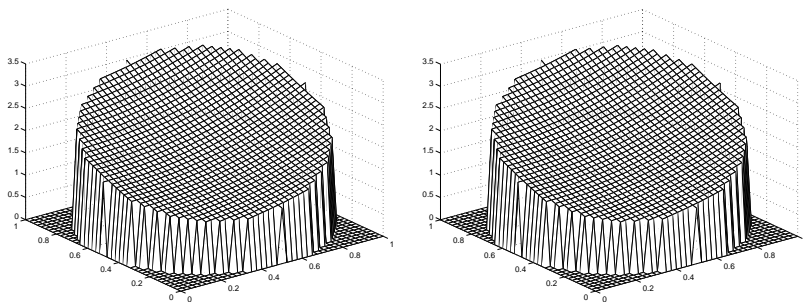


Figure 3: Profiles of u and $u_{n,m}$.

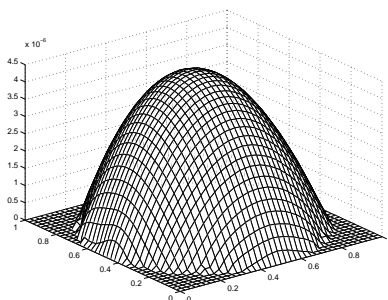


Figure 4: error

Table 3 shows that the exponential kernel provides more satisfactory numerical results.

Next we use the same methods in Example 2 for a different boundary value problem.

Example 3. Consider

$$\Delta u(x, y) = (4x^2 + 3)e^{x^2+y} - 9\sqrt{x^2 + y^2}, \quad (x, y) \in \Omega, \quad (75)$$

$$u(x, y) = e^{x^2+y} - (x^2 + y^2)^{\frac{3}{2}}, \quad (x, y) \in \partial\Omega, \quad (76)$$

where the exact solution is $u(x, y) = e^{x^2+y} - (x^2 + y^2)^{\frac{3}{2}}$. The boundary of Ω is an epitrochoid shape parameterized by $x(t) = 6 \cos t - \frac{1}{2} \cos(6t)$, $y(t) = 6 \sin t - \frac{1}{2} \sin(6t)$, $0 \leq t < 2\pi$, (see Fig.5). Let $f(x, y) = (4x^2 + 3)e^{x^2+y} - 9\sqrt{x^2 + y^2}$, and $f_\chi(x, y) = f(x, y) \cdot \chi(x, y)$. Use $\mathcal{B}_n f_\chi(x, y)$ as in Eq. (68) to approximate $f_\chi(x, y)$, and the particular solutions of $\Delta u = \mathcal{B}_n f_\chi$ are given as in Eq. (70). The kernel method is then applied, which we only use the exponential kernel. The approximation errors are shown in Table 4 with respect to different values of m , which shows that the numerical results do not improve for large values of m ,

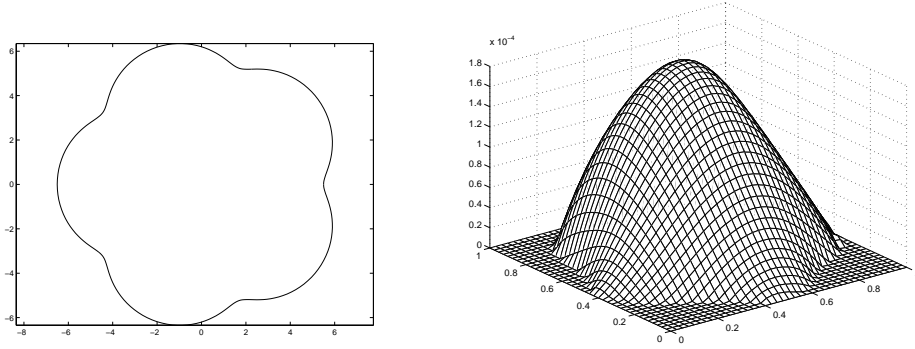


Figure 5: Graph of $\partial\Omega$ and error.

Table 4: $\|e\|_{L^\infty(\Omega)}$ on an arbitrary domain.

| m | $m = 12$ | $m = 16$ | $m = 20$ | $m = 24$ | $m = 30$ |
|-------------|--------------|---------------|---------------|---------------|---------------|
| kernel(exp) | $4.087e - 3$ | $3.3351e - 4$ | $1.7613e - 4$ | $1.7616e - 4$ | $1.7614e - 4$ |

Finally we present an example to compare kernel methods with Trefftz methods and MFS for a Poisson's equation.

Example 4. We consider

$$\Delta u(x, y) = (x^4 - y^4)e^{xy}, \quad (x, y) \in \Omega, \quad (77)$$

$$u(x, y) = (x^2 - y^2)e^{xy}, \quad (x, y) \in \partial\Omega. \quad (78)$$

where Ω is a peanut-shape region with the boundary parameterized by the polar form

$$r(t) = \sqrt{\cos 2t + \sqrt{1.1 - \sin^2 2t}}, \quad 0 \leq t \leq 2\pi \quad (79)$$

The exact solution is given by $u(x, y) = (x^2 - y^2)e^{xy}$. To evaluate the particular solution, the peanut shaped domain is embedded to a rectangular box $[-1, 1] \times [-0.5, 0.5]$. Let $f(x, y) = (x^4 - y^4)e^{xy}$, and $f_\chi(x, y)$ be given by

$$f_\chi(x, y) = f(x, y)V(-1, 1, 0.3; x)V(-0.5, 0.5, 0.3; y). \quad (80)$$

Use radial basis methods to find u_n as shown before. And then the kernel method, MFS and Trefftz method are correspondingly applied to solve

$$\Delta v(x, y) = 0, \quad (x, y) \in \Omega, \quad (81)$$

$$v(x, y) = (x^2 - y^2)e^{xy} - u_n(x, y), \quad (x, y) \in \partial\Omega, \quad (82)$$

as it was done in Example 1. To evaluate the approximation error, we use 40×20 equally distributed points in $[-1, 1] \times [-0.5, 0.5]$ and justify these points. The boundary of Ω with test points and also the profile of absolute errors are shown in Fig. 6. The approximation errors by three methods are presented in Table 5. It is clear to see that the kernel method performs better than the other two for large value of m .

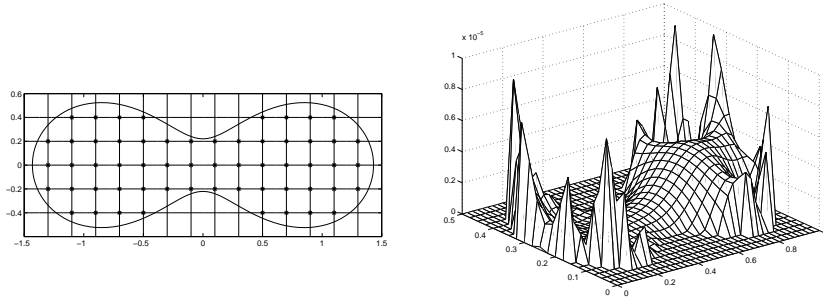


Figure 6: Graph of $\partial\Omega$ with test points and absolute error.

Table 5: $\|e\|_{L^\infty(\Omega)}$ on domain of peanut.

| m | $m = 12$ | $m = 16$ | $m = 20$ | $m = 24$ | $m = 30$ |
|-------------|---------------|---------------|---------------|---------------|---------------|
| kernel(exp) | $4.3855e - 4$ | $1.7613e - 4$ | $1.0898e - 4$ | $9.8481e - 6$ | $1.6051e - 5$ |
| MFS | $3.0080e - 4$ | $2.2129e - 4$ | $2.2e - 3$ | $2.232e - 3$ | $3.0465e - 3$ |
| Trefftz | $4.5649e - 4$ | $2.1845e - 4$ | $2.2571e - 3$ | $2.7064e - 2$ | $4.3313e - 2$ |

References

- [1] M.S. Abou-Dina, Implementation of Trefftz method for the solution of some elliptic boundary value problems, *Appl. Math. Comput.* 127 (2002) 125-147.
- [2] N. Aronszajn, Theory of reproducing kernels, *Transactions of the American Mathematical Society*, 68: 3 (1950) 337-404.
- [3] A. Bogomolny, Fundamental solutions method for elliptic boundary value problems, *SIAM. J. Numer. Anal.* 22 (4) (1985) 644-669.
- [4] G. Fairweather and A. Karageorghis, The method of fundamental solutions for elliptic boundary value problems, *Advances in Computational Mathematics* 9 (1998) 69-95.
- [5] M.A. Golberg, The method of fundamental solutions for Poisson's equation, *Engineering Analysis with Boundary Elements*, 16 (1995) 205-213.
- [6] I. Herrera, Trefftz method: A general theory, *Numer. Methods Partial Differ. Dqu.* 16 (6) (2000) 177-208.
- [7] X. Li, Radial basis approximation for Newtonian potentials, preprint.
- [8] X. Li, Radial basis approximation and its application to biharmonic equation, preprint.
- [9] X. Li, On convergence of the method of fundamental solutions for solving the Dirichlet problem of Poisson's equation, *Adv. Comput. Math.* 23 (2005) 265-277.
- [10] X. Li, Convergence of the method of fundamental solutions for Poisson's equation on the unit sphere, *Adv. Comput. Math.* (2006)
- [11] X. Li, On the method of fundamental solutions in R^2 , *Journal of Information & Computational Science* 4: 2 (2007) 567-586.
- [12] X. Li and C.S. Chen, A mesh free method using hyperinterpolation and fast Fourier transform for solving differential equations, *Engineering Analysis with Boundary Elements* 28 (2004) 1253-1260.
- [13] X. Li, C.H. Ho and C.S. Chen, Computational Test of Approximation of Functions and their derivatives by radial basis functions, *Neural Parallel Sci. Comput.* 10 (2002) 25-46.
- [14] X. Li and C.A. Micchelli, Approximation by radial bases and neural networks, *Numerical Algorithms* 25 (2000) 241-262.
- [15] W.R. Madych, An estimate for multivariate interpolation II, *Journal of Approximation Theory* 142 (2006) 116-128.
- [16] R. Schaback, Solving the Laplace equation by meshless collocation using harmonic kernels, preprint.
- [17] R. Schaback and H. Wendland, Kernel techniques: From machine learning to meshless methods, *Acta Numerica*, 15 (2006) 543-639.